Fiscal federalism or fiscal cooperation in a monetary union: what is preferable?

Hubert Kempf
Ecole Normale Supérieure Paris Saclay & CREST

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Ecole Normale Supérieure de Cachan, 61 Boulevard du Président Wilson, 92230 Cachan, France.

Abstract

In this paper we assess the relative merits of a fiscal federation and intergovernmental cooperation for managing fiscal policy in a monetary union. Using a standard macroeconomic model commonly used for macroeconomic policy exercises we prove that it is impossible to conclude that one solution is preferable to the other. A more potent federal fiscal instrument does not necessarily raise the interest of a fiscal federation. The welfare ranking of these two options depends on the variances of shocks and the various objectives of the policymakers. This results sheds light on the current development of the European monetary union.

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1 Introduction.

The current difficulties of the European monetary union following the debt crisis of the early 2010s and the painful discussion around the Greek debt have put to the fore the shortcomings of the initial institutional framework of this project. The necessity to complement the monetary side of the EMU by a stronger fiscal apparatus is contested by no one. However the precise contours of such apparatus are not agreed and hardly discussed.

In general economists are more likely to support the proposal of a fiscal federation flanking the EMU and the creation of a proper supranational or "federal" Treasury.[1] Political European leaders, likely expressing

[1]A recent proposal to be applied to the European Union (Wolff, 2012) is representative of this view. See also Bureau and Champsaur (1992) and Persson, Roland, Tabellini (1996) for standards expositions of the stakes of fiscal federalism applied to the European Union. The recent paper by Farhi and Werning (2012) has renewed the arguments in favor of a fiscal federal scheme complementing a currency union. See also the constitutional approach of Mueller (1997).

the feeling of their electorates, see such a step ahead with caution and prefer (or behave according to) an intergovernmental cooperation scheme.[2] This approach to the current woes of the EMU is seen with skepticism by academics and pundits alike.

The current paper aims at analyzing the debate between the advocates of these two options by means of an analytical comparison of their respective merits rooted in a simple yet formal macroeconomic model of a monetary union commonly used in policymaking studies. Two variants of the model are developed, one featuring a fiscal federalism scheme and the other one a cooperative approach to fiscal policy based on some agreement between the "country" or "national" policymakers.

The outcome supports an "agnostic" view on the problem: it is impossible to state without further precise explanations that a federation is to be prefered to an intergovernmental approach. It actually depends on the circumstances. More precisely, for a given economic structure of the union, what matters is the variances of the shocks and the differing objectives of the various policymakers active in the union. In other words, if we want to strenghten a monetary union by developing the fiscal side, we cannot dismiss either one of the two options we focus upon. In particular intergovernmental cooperation, setting aside the complexity of the bargaining process, cannot be dismissed as an unsound design of a monetary union. More troublesome, it may happen that a fiscal federalism scheme is dominated by a setting without any cooperation between fiscal players.

The issue at stake comes from the fact that a policy mix at the scale of the union is required in a monetary union. From a theoretical point of view, two arguments have been put forward to stress the heightened role of fiscal instruments in a monetary union. First the traditional argument is that the reduction in the number of monetary instrument compared to a multi-currency world with flexible exchange rates gives a larger role to the remaining, that is, fiscal instruments for stabilization purposes; a second more recent argument is that cross-border fiscal spillovers cannot be nullified by any monetary policy rule: insulation is impossible.[3] Empirically the evidence clearly shows the role of interjurisdictional public transfers cannot be ignored even though the magnitude of these spillovers is subject to discussion.[4]

Thus there is no disagreement on the need of overcoming the defects of fiscal policies when they are taken at the country level by national authorities which neglect or downplay the impact of their decisions on the rest of the union. The issue is about the proper design of the fiscal side of a monetary union.[5] Specifically, given the historical precedent of the United States, the discussion concentrates over the benefits of a fiscal federation.[6]

The model of a two-country monetary union allows for cross-border fiscal spillovers, distinguishing be-

[2]The recent book by Van Middelaar (2013) is an illuminating analysis of the workings of the European Council which embodies the intergovernmental cooperation approach.

[3]See Cooper, Peled and Kempf (2014).

[4]The seminal study is Asdrubali, Sorensen and Yosha (1996). Melitz (2004) provides a very detailed discussion of the empirical difficulties encountered in the attempts to quantify these spillovers. See also Evers (2006).

[5]Dixit and Lambertini (2003) have shown in a simple model, similar to the one set-up here, that the choice of a fiscal scheme can be irrelevant. Kempf and von Thadden (2014) have generalized this "symbiosis" result and proven that the required conditions are extremely restrictive.

[6]Several studies have developed this precedent with the aim of drawing lessons from this experience for Europe. See Bordo, Jonung and Martkiewicz (2011) and Henning and Kessler (2012).

tween the federal ones and the intercountry al ones. It includes two types of shocks: there is one nominal shock which affects identically the two countries and supply shocks which are country-specific and uncorrelated. Finally policymakers make their decision according to standard macroeconomic loss functions. We study three games which correspond to three different institutional variants: the case without a federal Treasury and no cooperation among fiscal authorities, which serves as a benchmark; the case without a federal Treasury but where fiscal authorities cooperate; and finally, the case with a federal Treasury materializing a fiscal federation but no cooperation among any policymakers. For each variant, we are able to compute the corresponding equilibrium and the implied expected losses for each policymaker. This allows us to compare from a normative point of view the three variants under study. We concentrate the comparison between these variants on the expected losses of the national fiscal authorities.

Formally our analysis relies on the undetermined coefficients method, commonly used to solve equations with rational expectations. Here we extend it to solve a system of interdependent equations with rational expectations. It could be used to solve more complex models than ours. The model studied here is simple and could easily be extended either in complexity with more policymakers or with a richer economic structure. A proper dynamic analysis including state variables would imply different game-theoretical tools.

2 The model.

We set up the model of a two-country monetary union where the two countries are endowed with autonomous fiscal authorities, and possibly a federal fiscal authority. There is a single central bank. Each fiscal national authority $i, i = 1, 2$, controls a fiscal instrument g_i. The federal Treasury when active controls its own instrument denoted by g_F. In the variants of the economy where this policymaker is absent (inactive), this instrument is set to 0.

Aggregate outputs.

The aggregate output level in country i, denoted by y_i, is given by the following equation:

$$y_i = \hat{y} + \alpha g_i + \beta g_j + \gamma g_F + b(\pi - \pi^e) + u_i \tag{1}$$

where $\hat{y}$ denotes the natural aggregate output level, equal in both countries s, π the inflation rate in the union, π^e the expected inflation rate rationally anticipated by private agents, and u_i the real shock affecting country i. Throughout the paper we consider $i, j = 1, 2; j \neq i$. u_i is an i.i.d. random variable of zero mean and variance σ_u^2. These shocks have the same distribution law in both countries but their realizations differ. α, β and γ measure the impacts on national output of fiscal impulses decided by the national fiscal authority, the other national authority and the federal authority, respectively. It is reasonable to assume that $\alpha > \beta$: the national fiscal policy is a stronger impact than the other country 's policy. This may be justified by stronger and more direct transmission channels. β measures the magnitude of trans-border spillovers. Similarly we assume $\beta \geq \gamma$: the national fiscal policy has more impact than the federal one, either because it is closer or because it is more efficient.

The (average) union's aggregate output level is equal to:

$$\overline{y} = \hat{y} + \frac{1}{2}\left(\alpha+\beta\right)\left(g_1+g_2\right) + \gamma g_F + b\left(\pi - \pi^e\right) + \bar{u} \tag{2}$$

where $\hat{y}$ is the (average) natural aggregate output level in the union, assumed to the same as the natural level in each country and $\bar{u}$ is the mean of the real shocks u_i.

Inflation.

Inflation at the union's level is given by the following equation:

$$\pi = c\left(g_1+g_2+g_F\right) + \pi_M + \varepsilon \tag{3}$$

where π_M represents the instrument controlled by the central bank of the union and ε the monetary shock affecting inflation. ε is an i.i.d. random variable of mean zero and variance σ_ε^2. In order to focus on the relationships between fiscal authorities, we assume in this chapter that π_M ie equal to 0, or equivalently that the union's central bank does not follow any active monetary policy rule. We maintain the assumption that the various fiscal instruments have the same impact on inflation.

There are no lagged terms in this economy. The endogenous variables of the model are thus function of the current realizations of shocks and of the objectives of the various policymakers. As far as policymaking is concerned, three options are possible, within the simple macromodel that we use:

1. the union is not a federation and there is no cooperation among policymakers. National fiscal authorities make their decision non-cooperatively.

2. the union is not a federation but national fiscal authorities cooperate and aim at minimizing a (weighted) sum of their losses. Here as we assume symmetrical countries we consider a simple sum of losses.

3. the union is a federation but there is no cooperation among policymakers. A federal Treasury actively controls its instrument and attempts to minimize a loss function the arguments of which are the union output gap and inflation. National fiscal authorities keep their autonomy and their control over their own fiscal instrument.

A trade-off emerges: a federation has the advantage of setting more policy instruments but does not encompass a cooperative scheme between any policymakers. Thus it is a priori impossible to state which option, federation or cooperation, is preferable.

To simplify the solutions of these variants and concentrate on fiscal policymakers, we assume that the central bank is inactive (does not respond to shocks) and set its monetary instrument π_m to 0.[7] We abstract from informational and fiscal policy implementation issues.

Loss functions.

[7] This assumption may be relaxed and active monetary policy rules can be introduced in the model.

The welfare criteria used by policymakers are simple and in line with the standard approach to macroeconomic policy. The various policymakers present in the monetary union aim at stabilizing output and inflation, however with different views on the ideal macroeconomic configuration. The two national fiscal authorities are concerned with national aggregate output, whereas the central bank and the federal Treasury worry about the union's aggregate output. All of them focus the union inflation rate. Moreover their output and inflation objectives may differ. Each authority wants to minimize the squared gap of output (country or union-wide) to a desired "natural" level as well as the squared value of inflation (meaning that their desired inflation rate is 0).

Formally, the welfare criterion of a policymakers is given by a quadratic loss function the arguments of which are the output gap and the inflation gap. These loss functions are the following:

Fiscal authority of country i:

$$L_i = \frac{1}{2}\theta\left(y_i - \tilde{\chi}\right)^2 + \frac{1}{2}\pi^2 \tag{4}$$

where $\tilde{\chi}$ is the level of union's aggregate output level desired by any national fiscal authority. θ expresses the relative weight given to the output objective compared to the inflation one by the national fiscal authorities.

Federal fiscal authority:

The loss function of the federal Treasury writes:

$$L_F = \frac{1}{2}\theta_F\left(\overline{y} - \tilde{\chi}_F\right)^2 + \frac{1}{2}\pi^2 \tag{5}$$

where $\tilde{\chi}_F$ is the union's aggregate output level desired by the fedral authority. We assume $\tilde{\chi}_F \neq \tilde{\chi}$. For simplicity reasons, we further assume: $\tilde{\chi}_F = 0$. Loosely speaking, we interpret in the following analyses $\tilde{\chi}$ as the objective gap between the federal and the national authorities. θ_F expresses the relative weight given to the output objective compared to the inflation one by the federal Treasury.

Union's monetary authority (Central bank):

The central bank's loss function is given by:

$$L_M = \frac{1}{2}\theta_M\left(\overline{y} - \tilde{\chi}_M\right)^2 + \frac{1}{2}\pi^2 \tag{6}$$

where $\tilde{\chi}_M$ is the level of union's aggregate output level desired by the monetary authority and $\overline{y}$ is the average union's aggregate product. θ_F expresses the relative weight given to the output objective compared to the inflation one by the central bank. Kempf and von Thadden (2014), generalizing a result obtained by Dixit and Lambertini (2003) have shown that such a model, with quadratic objective functions and linear relationships, makes cooperation between n players irrelevant as long as there are as many objectives as instruments, each player controls one instrument and their objective functions are identical as the symbiosis point (all variables at their desired values) is reached. Here we assume enough heterogeneity in the loss functions by means of the weights θ and the output objective χ to ensure that the various configurations we study do not generate symbiosis.

2.1 Options.

Three stages constitute the economy.

1. private agents form their expectations of inflation;
2. Shocks occur;
3. Policymakers make their decisions.

2.1.1 No fiscal cooperation.

We start with this benchmark case characterized by the absence of a federal Treasury and no cooperation between fiscal authorities. It is equivalent to assume that $g_F = 0$.

Country aggregate outputs are equal to:

$$y_i = \hat{y} + \alpha g_i + \beta g_{-i} + b\left(\pi - \pi^e\right) + u_i$$

and the union's aggregate output:

$$\overline{y}_n^* = \hat{y} + \frac{1}{2}\left(\alpha + \beta\right)\left(g_1 + g_2\right) + b\left(\pi - \pi^e\right) + \bar{u}.$$

We define $\chi = \tilde{\chi} - \hat{y}$ and we refer to as the "objective gap" of fiscal authorities. We denote by g_{in}^* the optimal decision of the fiscal authority in country i. Solving this game generates the following reduced forms:

$$g_{in}^* = \frac{\theta\left(\alpha + bc\right)}{\theta\left(\alpha + bc\right)\left(\alpha + \beta + 2bc\right) + 2c^2}\chi - \frac{\left(c + \theta b\left(\alpha + bc\right)\right)}{\theta\left(\alpha + bc\right)\left(\alpha + \beta + 2bc\right) + 2c^2}\varepsilon$$

$$-\frac{\left(\theta\left(\alpha + bc\right)^2 + c^2\right)}{\left[\theta\left(\alpha + bc\right)\left(\alpha + \beta + 2bc\right) + 2c^2\right]\left(\alpha - \beta\right)}u_i + \frac{\left(\theta\left(\beta + bc\right)\left(\alpha + bc\right) + c^2\right)}{\left[\theta\left(\alpha + bc\right)\left(\alpha + \beta + 2bc\right) + 2c^2\right]\left(\alpha - \beta\right)}u_j. \tag{7}$$

The resulting inflation, which we denote by π_n^*, is equal to:

$$\pi_n^* = \frac{\theta\left(\alpha + bc\right)}{\theta\left(\alpha + bc\right)\left(\alpha + \beta\right) + 2c^2}\left[2c\chi - c\left(u_1 + u_2\right) + \left(\alpha + \beta\right)\varepsilon\right]. \tag{8}$$

The reduced forms for national outputs are:

$$y_{in}^* = \hat{y} + \frac{\theta\left(\alpha + bc\right)\left(\alpha + \beta\right)}{\theta\left(\alpha + bc\right)\left(\alpha + \beta\right) + 2c^2}\chi - \frac{\left(\alpha + \beta\right)c}{\theta\left(\alpha + bc\right)\left(\alpha + \beta + 2bc\right) + 2c^2}\varepsilon$$

$$+\frac{c^2}{\theta\left(\alpha + bc\right)\left(\alpha + \beta + 2bc\right) + 2c^2}\left(u_1 + u_2\right) \tag{9}$$

and thus equal to the union's aggregate (average) output level $\overline{y}_n^*$. The national fiscal authorities cannot compensate entirely the impact of shocks in the aggregate output, nor on inflation. The impact of real shocks is positive and identical in both countries.

2.1.2 Fiscal cooperation.

This variant corresponds to intergovernmental cooperation, in the absence of a federation, that is, without a federal Treasury enpowered with a fiscal instrument and macroeconomic objectives. The structure of the economy is the same as before.

In these circumstances, intercountry cooperation seeks to minimize the unweighted sum of the losses incurred by the two cooperating players. The assumption of an unweighted sum is consistent with the symmetry of the two countries and the absence of size effects in this model. Formally the loss function in this case is:

$$L_C = \frac{1}{2}\left\{\frac{\theta}{2}\left[(y_1-\tilde{\chi})^2+(y_2-\tilde{\chi})^2\right]\right\}+\frac{1}{2}\pi^2 \tag{10}$$

The two fiscal authorities set cooperatively the levels of the two fiscal instruments, g_1 et g_2, in order to minimize this joint loss. We denote g_{iC}^* the optimal decision for country i in a cooperative setting.

The solutions of this minimization program are:

$$g_{iC}^* = \frac{\frac{\theta}{2}(\alpha+\beta+2bc)}{\left[\frac{\theta}{2}(\alpha+\beta+2bc)(\alpha+\beta)+2c^2\right]}\chi - \frac{\left[c+\frac{\theta}{2}b(\alpha+\beta+2bc)\right]}{\left[\frac{\theta}{2}(\alpha+\beta+2bc)^2+2c^2\right]}\varepsilon$$

$$-\frac{\frac{\theta}{2}(\alpha+\beta+2bc)(\alpha+bc)+c^2}{(\alpha-\beta)\left[\frac{\theta}{2}(\alpha+\beta+2bc)^2+2c^2\right]}u_i + \frac{\frac{\theta}{2}(\alpha+\beta+2bc)(\beta+bc)+c^2}{(\alpha-\beta)\left[\frac{\theta}{2}(\alpha+\beta+2bc)^2+2c^2\right]}u_j \tag{11}$$

The corresponding inflation is given by the following equation:

$$\pi_C^* = \frac{\theta c(\alpha+\beta+2bc)}{\left[\frac{\theta}{2}(\alpha+\beta+2bc)(\alpha+\beta)+2c^2\right]}\chi + \frac{\left[\frac{\theta}{2}(\alpha+\beta+2bc)\right](\alpha+\beta)}{\left[\frac{\theta}{2}(\alpha+\beta+2bc)^2+2c^2\right]}\varepsilon$$

$$-\frac{\frac{\theta}{2}c(\alpha+\beta+2bc)}{\frac{\theta}{2}(\alpha+\beta+2bc)^2+2c^2}(u_1+u_2) \tag{12}$$

Country aggregate outputs are equal to:

$$y_{iC}^* = \hat{y} + \frac{\theta c(\alpha+\beta+2bc)(\alpha+\beta)}{\left[\frac{\theta}{2}(\alpha+\beta+2bc)(\alpha+\beta)+2c^2\right]}\chi - \frac{(\alpha+\beta)c}{\left[\frac{\theta}{2}(\alpha+\beta+2bc)^2+2c^2\right]}\varepsilon$$

$$+\frac{c^2}{\left[\frac{\theta}{2}(\alpha+\beta+2bc)^2+2c^2\right]}(u_1+u_2). \tag{13}$$

As they are equal, they are equal to the union's output. This is consistent with the logic of cooperation: shocks are managed taking into account the cross-border effects of policy decisions.

2.1.3 Fiscal federalism.

Fiscal federalism is based on an active federal Treasury. The objectives of this treasury, as formalized by the loss function (5) are union-wide. Yet the three fiscal authorities act non-cooperatively and simultaneously (the central bank is still assumed to be inactive). We look for the solution to this three-player Nash game.

We denote by g^*_{iF} the optimal decision of the fiscal authority in country i in the presence of fiscal federalism and g^*_{FF} the optimal decision of the federal Treasury.

The outcomes of the Nash equilibrium of this game are:

$$
\begin{aligned}
g^*_{iF} = & \frac{(\alpha+bc)\left(\theta\gamma(\gamma+bc)+c^2\right)}{c^2(\gamma-\alpha)(2\gamma-\alpha-\beta)}\chi - \frac{\gamma}{c\left[2\gamma-\alpha-\beta\right]}\varepsilon \\
& + \frac{(\alpha-\gamma)}{(2\gamma-\alpha-\beta)(\alpha-\beta)}u_i + \frac{(\gamma-\beta)}{(\alpha-\beta)(2\gamma-\alpha-\beta)}u_j
\end{aligned}
\tag{14}
$$

$$
\begin{aligned}
g^*_{FF} = & -\frac{\left(2c^2+\theta(\gamma+bc)(\alpha+\beta)\right)(\alpha+bc)}{c^2(\gamma-\alpha)(2\gamma-\alpha-\beta)}\chi + \frac{(\alpha+\beta)}{c(2\gamma-\alpha-\beta)}\varepsilon \\
& - \frac{1}{2\gamma-\alpha-\beta}(u_1+u_2)
\end{aligned}
\tag{15}
$$

Inflation, denoted by π^*_F, is given by:

$$\pi^*_F = \frac{\theta(\alpha+bc)(\gamma+bc)}{c(\gamma-\alpha)}\chi \tag{16}$$

In this setting, the impacts of shocks on inflation are entirely annihilated. The inflation bias depends on the extent of the disagreement between fiscal authorities.

In these conditions, aggregate national outputs are equal to:

$$y^*_{iF} = \hat{y} - \frac{(\alpha+bc)}{(\gamma-\alpha)}\chi - \frac{2\gamma\beta}{(2\gamma-\alpha-\beta)(\alpha-\beta)}u_i \tag{17}$$

The foreign real shock as well as the monetary shock have no impact on domestic aggregate output. The impact of the domestic real shock is the same in both countries, in conformity to the symmetry assumption between the two countries.

Lastly the union's aggregate output level which is denoted by $\overline{y}^*_F$ is equal to:

$$\overline{y}^*_F = \hat{y} - \frac{(\alpha+bc)}{(\gamma-\alpha)}\chi - \frac{2\gamma\beta}{(2\gamma-\alpha-\beta)(\alpha-\beta)}\overline{u} \tag{18}$$

As expected, the monetary shock does not impact outputs since it has no impact on inflation.

3 Comparing outcomes.

The previous formulas allow us to compare the macroeconomic outcomes of the three variants of monetary union.

3.1 Cooperation vs no cooperation.

We compare the outcome of the union characterized by cooperation among fiscal authorities with the outcome of the union with such a cooperation.

For inflation we get:

$$\pi^*_C - \pi^*_n = \left[\frac{\theta c(\alpha+\beta+2bc)\left[\theta(\alpha+bc)(\alpha+\beta)+2c^2\right] - 2c\theta(\alpha+bc)\left[\frac{\theta}{2}(\alpha+\beta+2bc)(\alpha+\beta)+2c^2\right]}{\left[\theta(\alpha+bc)(\alpha+\beta)+2c^2\right]}\right]\chi$$

$$-\left[\frac{\frac{\theta}{2}c\left(\alpha+\beta+2bc\right)\left[\theta\left(\alpha+bc\right)\left(\alpha+\beta\right)+2c^{2}\right]-\left[\frac{\theta}{2}\left(\alpha+\beta+2bc\right)^{2}+2c^{2}\right]\theta c\left(\alpha+bc\right)}{\left[\frac{\theta}{2}\left(\alpha+\beta+2bc\right)^{2}+2c^{2}\right]}\right]\left(u_{1}+u_{2}\right)$$

$$+\left[\frac{\left[\frac{\theta}{2}\left(\alpha+\beta+2\mathrm{bc}\right)\right]\left[\theta\left(\alpha+\mathrm{bc}\right)\left(\alpha+\beta\right)+2\mathrm{c}^{2}\right]-\theta\left(\alpha+\mathrm{bc}\right)\left[\frac{\theta}{2}\left(\alpha+\beta+2\mathrm{bc}\right)^{2}+2\mathrm{c}^{2}\right]}{\left[\frac{\theta}{2}\left(\alpha+\beta+2\mathrm{bc}\right)^{2}+2\mathrm{c}^{2}\right]\left[\theta\left(\alpha+\mathrm{bc}\right)\left(\alpha+\beta\right)+2\mathrm{c}^{2}\right]}\right]\left(\alpha+\beta\right)\varepsilon \qquad (19)$$

We notice that the various terms in brackets are all negative when $\alpha > \beta$. In particular, the inflation bias (the coefficient associated with χ) is weaker in the case of cooperation than in the case of no cooperation. And the variance of inflation is smaller in the former case than in the latter.

For national aggregate output, we get:

$$y_C^* - y_{in}^* = -\left[\frac{\frac{\theta}{2}\left(\alpha+\beta+2bc\right)\left[\alpha-\beta\right]}{\left[\frac{\theta}{2}\left(\alpha+\beta+2bc\right)^{2}+2c^{2}\right]\left[\theta\left(\alpha+bc\right)\left(\alpha+\beta+2bc\right)+2c^{2}\right]}\right]\left(c^{2}\left(u_{1}+u_{2}\right)-\left(\alpha+\beta\right)c\varepsilon\right) \qquad (20)$$

The inflation bias is the same in both variants; thus it does not enter this difference. The term in brackets is negative for $\alpha > \beta$. Hence the variance of output is smaller in the case of cooperation than in the case of no-cooperation.

In brief, this is what was expected: cooperation reduces the inflation bias as well as the variance of output.

3.2 Comparing cooperation and federalism.

We proceed similarly for the comparison between a union based on cooperation and one based on fiscal federalism.

For inflation we get:

$$\pi_C^* - \pi_F^* = \left[\frac{\theta c\left(\alpha+\beta+2bc\right)c\left(\gamma-\alpha\right)-\theta\left(\alpha+bc\right)\left(\gamma+bc\right)\left[\frac{\theta}{2}\left(\alpha+\beta+2bc\right)\left(\alpha+\beta\right)+2c^{2}\right]}{\left[\frac{\theta}{2}\left(\alpha+\beta+2bc\right)\left(\alpha+\beta\right)+2c^{2}\right]c\left(\gamma-\alpha\right)}\right]\chi$$

$$-\left[\frac{\frac{\theta}{2}c\left(\alpha+\beta+2bc\right)}{\frac{\theta}{2}\left(\alpha+\beta+2bc\right)^{2}+2c^{2}}\right]\left(u_{1}+u_{2}\right)+\left[\frac{\left[\frac{\theta}{2}\left(\alpha+\beta+2\mathrm{bc}\right)\right]\left(\alpha+\beta\right)}{\left[\frac{\theta}{2}\left(\alpha+\beta+2\mathrm{bc}\right)^{2}+2\mathrm{c}^{2}\right]}\right]\left(\alpha+\beta\right)\varepsilon \qquad (21)$$

The first term in brackets is positive for $\alpha > \gamma$; the other terms are always positive. The inflation bias is bigger in a cooperating union than in a federation and the variance of inflation is higher.

For national aggregate output, we get:

$$y_C^* - y_{iF}^* = \left[\frac{\left(\alpha+bc\right)\left[\theta\left(\alpha+\beta\right)\left(\gamma-2\alpha-bc\right)-2c^{2}\right]}{\left(\gamma-\alpha\right)}\right]\chi+\frac{\left(\alpha+\beta\right)c}{\left[\frac{\theta}{2}\left(\alpha+\beta+2bc\right)^{2}+2c^{2}\right]}\varepsilon$$

$$-\left[\frac{c^{2}\left(2\gamma-\alpha-\beta\right)\left(\alpha-\beta\right)-\theta\gamma\beta\left(\alpha+\beta+2bc\right)^{2}-4\gamma\beta c^{2}}{\left[\frac{\theta}{2}\left(\alpha+\beta+2bc\right)^{2}+2c^{2}\right]\left(2\gamma-\alpha-\beta\right)\left(\alpha-\beta\right)}\right]u_{i}+\frac{2\gamma\beta}{\left(2\gamma-\alpha-\beta\right)\left(\alpha-\beta\right)}c^{2}u_{j} \qquad (22)$$

The sign of the first term in brackets is ambiguous even when $\alpha > \gamma$, as well as of the third one. The second term is positive as well as the fourth one (when assuming that $\beta \geq \gamma$). The expected difference in output gaps is thus ambiguous.

For the union's aggregate output, the difference is:

$$y_C^* - \overline{y}_F^* = \frac{(\alpha + bc)\left(\theta(\alpha+\beta)(\gamma - 2\alpha - bc) - 2c^2\right)}{\left(\theta(\alpha+bc)(\alpha+\beta) + 2c^2\right)(\gamma - \alpha)}\chi + \frac{(\alpha+\beta)c}{\left[\frac{\theta}{2}(\alpha+\beta+2bc)^2 + 2c^2\right]}\varepsilon$$
$$-\frac{c^2(2\gamma - \alpha - \beta)(\alpha - \beta) - \gamma\beta\left(\frac{\theta}{2}(\alpha+\beta+2bc)^2 + 2c^2\right)}{\left[\frac{\theta}{2}(\alpha+\beta+2bc)^2 + 2c^2\right](2\gamma - \alpha - \beta)(\alpha - \beta)}(u_1 + u_2) \tag{23}$$

The sign of the firstand third terms in brackets is ambiguous even when $\alpha > \gamma$. The expected difference in output gaps is thus ambiguous.

4 Expected losses.

In order to compare the three variants of a monetary union that we selected, we use as criteria the expected losses of the various policymakers. We concentrate on the comparison between the cooperative union and the union combined with a fiscal federation. The non-cooperative solution is solely developed as a benchmark. We focus our discussion on the expected losses of national fiscal players. As neither monetary policy nor the policy mix are our subject of study, we do not elaborate on the expected loss of the central bank. We conjecture that the preferences of the national fiscal players are representative of the preferences of the national governments, or of the national political bodies because these players emanate from their government or their constituency. Thus at a constitutional stage the choice (whatever the procedure used to choose an institutional setting for a monetary union) are likely to be based on welfare criteria similar to the loss functions used by national fiscal authorities. On the other hand, the central bank and possibly the federal Treasury can be seen as agencies and thus not properly backed by a political constituency with constitutional powers.

4.1 Options

4.1.1 No cooperation.

When national fiscal authorities do not cooperate, expected losses are egal to:

- for national fiscal authorities:

$$E\left(L_{in}^*\right)^2 = A_{in}^*\chi^2 + B_{in}^*\sigma_u^2 + C_{in}^*\sigma_\varepsilon^2 \tag{24}$$

  with

$$A_{in}^* = \frac{2\theta c^2\left(c^2 + \theta(\alpha+bc)^2\right)}{\left(\theta(\alpha+bc)(\alpha+\beta) + 2c^2\right)^2}, \quad B_{in}^* = \frac{\theta c^2\left(c^2 + \theta(\alpha+bc)^2\right)}{\left(\theta(\alpha+bc)(\alpha+\beta+2bc) + 2c^2\right)^2}$$

$$C_{in}^* = \frac{\theta c^2\left((\alpha+\beta)^2 + \theta(\alpha+bc)^2\right)}{\left(\theta(\alpha+bc)(\alpha+\beta+2bc) + 2c^2\right)^2}$$

- for the central bank

$$E\left(L_{Mn}^{*}\right)^{2}=A_{Mn}^{*}\chi^{2}++B_{Mn}^{*}\sigma_{u}^{2}+C_{Mn}^{*}\sigma_{\varepsilon}^{2} \tag{25}$$

with

$$A_{Mn}^{*}=\frac{1}{2}\frac{\theta^{2}\left(\alpha+bc\right)^{2}\left(\theta_{M}\left(\alpha+\beta\right)+4c^{2}\right)}{\left(\theta\left(\alpha+bc\right)\left(\alpha+\beta\right)+2c^{2}\right)^{2}},\quad B_{Mn}^{*}=\frac{c^{2}\left(\theta_{M}c^{2}+\theta^{2}\left(\alpha+bc\right)^{2}\right)}{\left(\theta\left(\alpha+bc\right)\left(\alpha+\beta+2bc\right)+2c^{2}\right)^{2}}$$

$$C_{Mn}^{*}=\frac{\left(\alpha+\beta\right)^{2}\left(\theta_{M}c^{2}+\theta^{2}\left(\alpha+bc\right)^{2}\right)}{\left(\theta\left(\alpha+bc\right)\left(\alpha+\beta+2bc\right)+2c^{2}\right)^{2}}$$

4.1.2 Fiscal cooperation.

On the other hand, when national fiscal authorities do cooperate, the expected loss for a national fiscal authority is equal to:

$$E\left(L_{iC}^{*}\right)^{2}=A_{iC}^{*}\chi^{2}+B_{iC}^{*}\sigma_{u}^{2}+C_{iC}^{*}\sigma_{\varepsilon}^{2} \tag{26}$$

with

$$A_{iC}^{*}=\frac{1}{2}\theta\frac{\left(\frac{\theta}{2}\left(\alpha+\beta+2bc\right)\left(\alpha+\beta\right)-2c^{2}\right)^{2}+\theta c^{2}\left(\alpha+\beta+2bc\right)^{2}}{\left(\frac{\theta}{2}\left(\alpha+\beta+2bc\right)\left(\alpha+\beta\right)+2c^{2}\right)^{2}}$$

$$B_{iC}^{*}=\frac{c^{2}\left(\left(\frac{\theta}{2}\right)^{2}\left(\alpha+\beta+2bc\right)^{2}+\theta c^{2}\right)}{\left(\frac{\theta}{2}\left(\alpha+\beta+2bc\right)^{2}+2c^{2}\right)^{2}},\quad C_{iC}^{*}=\frac{1}{2}\frac{\left(\left(\frac{\theta}{2}\right)^{2}\left(\alpha+\beta+2bc\right)^{2}+\theta c^{2}\right)\left(\alpha+\beta\right)}{\left(\frac{\theta}{2}\left(\alpha+\beta+2bc\right)^{2}+2c^{2}\right)^{2}}$$

Both A_{iC}^{*} and B_{iC}^{*} are ambiguous function of β (see appendix). They evidently do not depend on γ. Therefore expected losses of national fiscal authorities depend ambiguously on the fiscal spillover magnitude. Here the two national authorities are able to internalize the cross-border fiscal spillovers thanks to their cooperation which is an improvement over the no-cooperation outcome. However they manipulate only two instruments whereas there are three shocks to manage. The three shocks are thus affecting the expected losses of all policymakers.

The expected loss for the central bank is equal to:

$$E\left(L_{MC}^{*}\right)^{2}=A_{MC}^{*}\chi^{2}+B_{MC}^{*}\sigma_{u}^{2}+C_{MC}^{*}\sigma_{\varepsilon}^{2} \tag{27}$$

with

$$A_{MC}^{*}=\frac{1}{2}\frac{\theta^{2}c^{2}\left(\alpha+\beta+2bc\right)^{2}}{\left[\frac{\theta}{2}\left(\alpha+\beta+2bc\right)\left(\alpha+\beta\right)+2c^{2}\right]^{2}}\left(\theta_{M}\left(\alpha+\beta\right)^{2}+1\right)$$

$$B_{MC}^{*}=\frac{c^{2}\left(\left(\frac{\theta}{2}\right)^{2}\left(\alpha+\beta+2bc\right)^{2}+\theta_{M}c^{2}\right)}{\left[\frac{\theta}{2}\left(\alpha+\beta+2bc\right)^{2}+2c^{2}\right]^{2}}\quad C_{MC}^{*}=\frac{1}{2}\frac{\left(\alpha+\beta\right)^{2}c^{2}\left(\left(\frac{\theta}{2}\right)^{2}\left(\alpha+\beta+2bc\right)^{2}+\theta_{M}c^{2}\right)}{\left[\frac{\theta}{2}\left(\alpha+\beta+2bc\right)^{2}+2c^{2}\right]^{2}}$$

4.1.3 Fiscal federalism.

When a federal Treasury is active, the expected losses are equal to:

- for the national fiscal authorities:

$$E\left(L_{iF}^{*}\right)^{2}=A_{iF}^{*}\chi^{2}+B_{iF}^{*}\sigma_{u}^{2} \tag{28}$$

with

$$A_{iF}^{*}=\frac{1}{2}\frac{\theta\left(\gamma+bc\right)^{2}\left(c^{2}+\theta\left(\alpha+bc\right)^{2}\right)}{c^{2}\left(\gamma-\alpha\right)^{2}},\quad B_{iF}^{*}=\theta\frac{2\gamma^{2}\beta^{2}}{\left(2\gamma-\alpha-\beta\right)^{2}\left(\alpha-\beta\right)^{2}}$$

We get that (See appendix):

$$\frac{\partial A_{iF}^{*}}{\partial\gamma}=\frac{\theta\left(\gamma+bc\right)\left(c^{2}+\theta\left(\alpha+bc\right)^{2}\right)\left(\alpha+bc\right)}{c^{2}\left(\alpha-\gamma\right)^{3}}>0$$

$$\frac{\partial B_{iF}^{*}}{\partial\gamma}=-\frac{4\theta\gamma\beta^{2}\left(\alpha+\beta\right)}{\left(2\gamma-\alpha-\beta\right)^{2}\left(\alpha-\beta\right)^{2}}<0.$$

and

$$\frac{\partial A_{iF}^{*}}{\partial\beta}=0$$

$$\frac{\partial B_{iF}^{*}}{\partial\beta}==-4\gamma^{2}\beta\theta\left(\frac{\alpha^{2}+\beta^{2}-2\gamma\alpha}{\left(2\gamma-\alpha-\beta\right)^{2}\left(\alpha-\beta\right)^{2}}\right)<0.$$

The advantage of this variant of a monetary union is that the presence of a federal Treasury adds an policy instrument: the federal fiscal one. This allows to eliminate the impact of the nominal shock ε. However these policymaker do not cooperate, hence cross-border spillovers are not internalized and generate a suboptimal outcome.

Expected losses for national fiscal authorities are increasing functions of θ and χ, decreasing functions of β and depend ambiguously on γ. Expected losses are the higher when the impacts of fiscal instruments are closer. When the difference $(\alpha-\gamma)$ is very small, the impact of the objective gap is very large. When the difference $(\alpha-\gamma)$ or when the difference $(2\gamma-\alpha-\beta\approx 0)$ are very small, the impact of the supply shock variance is magnified. This is due to the fact that the spillovers are very large and not well taken into account because of the absence of cooperation.

- for the central bank:

$$E\left(L_{MF}^{*}\right)^{2}=A_{MF}^{*}\chi^{2}+B_{MF}^{*}\sigma_{u}^{2} \tag{29}$$

with

$$A_{MF}^{*}=\frac{1}{2}\frac{\left(\alpha+bc\right)^{2}\left(\theta_{M}c^{2}+\theta^{2}\left(\gamma+bc\right)^{2}\right)}{c^{2}\left(\gamma-\alpha\right)^{2}},\quad B_{MF}^{*}=\frac{2\theta_{M}\gamma^{2}\beta^{2}}{\left(2\gamma-\alpha-\beta\right)^{2}\left(\alpha-\beta\right)^{2}}$$

- for the federal fiscal authority:

$$E\left(L_{FF}^{*}\right)^{2}=A_{FF}^{*}\chi^{2}+B_{FF}^{*}\sigma_{u}^{2} \tag{30}$$

with

$$A_{FF}^{*}=\frac{1}{2}\frac{\left(\alpha+bc\right)^{2}\left(\theta_{F}c^{2}+\theta^{2}\left(\gamma+bc\right)^{2}\right)}{c^{2}\left(\gamma-\alpha\right)^{2}},\quad B_{FF}^{*}=\frac{1}{2}\theta_{F}\left(\frac{2\gamma\beta}{\left(2\gamma-\alpha-\beta\right)\left(\alpha-\beta\right)}\right)^{2}\sigma_{u}^{2}$$

The following tables summarize the results obtained for the various variants we study.

Expected losses

	Fiscal authority i
No cooperation	$\frac{2\theta c^2\left(c^2+\theta(\alpha+bc)^2\right)}{(\theta(\alpha+bc)(\alpha+\beta)+2c^2)^2}\chi^2 + \frac{\theta c^2\left(c^2+\theta(\alpha+bc)^2\right)}{(\theta(\alpha+bc)(\alpha+\beta+2bc)+2c^2)^2}\sigma_u^2 + \frac{1}{2}\frac{\theta c^2\left((\alpha+\beta)^2+\theta(\alpha+bc)^2\right)}{(\theta(\alpha+bc)(\alpha+\beta+2bc)+2c^2)^2}\sigma_\varepsilon^2$
Fiscal cooperation	$\frac{1}{2}\theta\frac{\left(\frac{\theta}{2}(\alpha+\beta+2bc)(\alpha+\beta)-2c^2\right)^2+\theta c^2(\alpha+\beta+2bc)^2}{\left(\frac{\theta}{2}(\alpha+\beta+2bc)(\alpha+\beta)+2c^2\right)^2}\chi^2 + \frac{c^2\left(\left(\frac{\theta}{2}\right)^2(\alpha+\beta+2bc)^2+\theta c^2\right)}{\left(\frac{\theta}{2}(\alpha+\beta+2bc)^2+2c^2\right)^2}\sigma_u^2 + \frac{1}{2}\frac{\left(\left(\frac{\theta}{2}\right)^2(\alpha+\beta+2bc)^2+\theta c^2\right)(\alpha+\beta)}{\left(\frac{\theta}{2}(\alpha+\beta+2bc)^2+2c^2\right)^2}\sigma_\varepsilon^2$
Fiscal federalism	$\frac{1}{2}\frac{\theta(\gamma+bc)^2\left(c^2+\theta(\alpha+bc)^2\right)}{c^2(\gamma-\alpha)^2}\chi^2+\theta\frac{2\gamma^2\beta^2}{(2\gamma-\alpha-\beta)^2(\alpha-\beta)^2}\sigma_u^2$

	Central bank
No cooperation	$\frac{1}{2}\frac{\theta^2(\alpha+bc)^2\left(\theta_M(\alpha+\beta)+4c^2\right)}{(\theta(\alpha+bc)(\alpha+\beta)+2c^2)^2}\chi^2 + \frac{\theta c^2\left(c^2+\theta(\alpha+bc)^2\right)}{(\theta(\alpha+bc)(\alpha+\beta+2bc)+2c^2)^2}\sigma_u^2 + \frac{1}{2}\frac{\theta c^2\left((\alpha+\beta)^2+\theta(\alpha+bc)^2\right)}{(\theta(\alpha+bc)(\alpha+\beta+2bc)+2c^2)^2}\sigma_\varepsilon^2$
Fiscal cooperation	$\frac{1}{2}\frac{\theta^2c^2(\alpha+\beta+2bc)^2}{\left[\frac{\theta}{2}(\alpha+\beta+2bc)(\alpha+\beta)+2c^2\right]^2}\left(\theta_M(\alpha+\beta)^2+1\right)\chi^2 + \frac{c^2\left(\theta_M c^2+\left(\frac{\theta}{2}\right)^2(\alpha+\beta+2bc)^2\right)}{\left[\frac{\theta}{2}(\alpha+\beta+2bc)^2+2c^2\right]^2}\sigma_u^2 + \frac{1}{2}\frac{(\alpha+\beta)^2c^2\left(\theta_M c^2+\left(\frac{\theta}{2}\right)^2(\alpha+\beta+2bc)^2\right)}{\left[\frac{\theta}{2}(\alpha+\beta+2bc)^2+2c^2\right]^2}\sigma_\varepsilon^2$
Fiscal federalism	$\frac{1}{2}\frac{(\alpha+bc)^2\left(\theta_M c^2+\theta^2(\gamma+bc)^2\right)}{c^2(\gamma-\alpha)^2}\chi^2 + \frac{2\theta_M\gamma^2\beta^2}{(2\gamma-\alpha-\beta)^2(\alpha-\beta)^2}\sigma_u^2$

4.2 Comparing expected losses.

The complexity of the formulas obtained above, despite the simplicity of the model we use, makes clear that the comparison is not immediate. The key elements of this comparison are the variances of shocks, the diverging targets of the policymakers and the characteristics of the inflation-output trade-off, which depend on the institutional framework of the monetary union.

More precisely, from (26) and (28), we get:

$$E\left(L_{iF}^{*}\right)^2 - E\left(L_{iC}^{*}\right)^2 = \left(A_{iF}^{*} - A_{iC}^{*}\right)\chi^2 + \left(B_{iF}^{*} - B_{iC}^{*}\right)\sigma_u^2 - C_{iC}^{*}\sigma_\varepsilon^2.$$

It is a priori impossible to give the sign of this difference. A corollary is that a fiscal federation is not necessarily Pareto-superior to a fiscal cooperation scheme. This is the main result of our analysis: it is impossible to state that a monetary union is viable solely if complemented by a fiscal federation with an active federal Treasury contributing to macroeconomic stabilization. Cooperation is preferable to a setting based on a fiscal federation for some configurations of shocks and objectives.

Given the impact of γ on the various coefficients in this equation (see above), this difference is an ambiguous function of the federal fiscal spillover coefficient. In other words, a more potent federal fiscal tool does not imply that the welfare of national fiscal authorities under fiscal federalism is increased. This counter-intuitive result comes from the non-cooperating behaviors of the national fiscal authorities.

More precisely we get from previous equation:

$$E\left(L_{iF}^{*}\right)^{2}<E\left(L_{iC}^{*}\right)^{2}\Leftrightarrow\sigma_{u}^{2}<\frac{C_{iC}^{*}\sigma_{\varepsilon}^{2}-\left(A_{iF}^{*}-A_{iC}^{*}\right)\chi^{2}}{\left(B_{iF}^{*}-B_{iC}^{*}\right)}. \tag{31}$$

As one achievement of a fiscal federation is to eliminate the impact of the nominal shock thanks to the presence of an additional instrument, such a design is the more valuable, the higher is the variance of this shock. The two institutional frameworks are equivalent for national fiscal authorities (in the sense they generate the same expected loss) for pairs $\left(\sigma_u^2,\sigma_\varepsilon^2\right)$ satisfying the following equation:

$$\sigma_{u}^{2}=\frac{C_{iC}^{*}\sigma_{\varepsilon}^{2}-\left(A_{iF}^{*}-A_{iC}^{*}\right)\chi^{2}}{\left(B_{iF}^{*}-B_{iC}^{*}\right)}.$$

This corresponds to a line splitting the positive orthant $\left(\sigma_u^2,\sigma_\varepsilon^2\right)$ in two countries. Its slope is equal to $\frac{C_{iC}^{*}}{B_{iF}^{*}-B_{iC}^{*}}$, positive or negative. We get from above:

$$B_{iF}^{*}-B_{iC}^{*}=\frac{2\theta\gamma^{2}\beta^{2}}{\left(2\gamma-\alpha-\beta\right)^{2}\left(\alpha-\beta\right)^{2}}-\frac{c^{2}\frac{\theta}{2}}{\left(\frac{\theta}{2}\left(\alpha+\beta+2bc\right)^{2}+2c^{2}\right)^{2}}$$

This difference is negative if the spillover parameters (β and γ) are small and positive if they are large and close to α.

For pairs posited above this line, cooperation is preferable to federation (since the aim is to minimize expected losses). In other words, an increase in the variance of real shocks relative to the variance of nominal shock, makes cooperation relatively more enticing. But it is not so much the nature of the shocks which explains this result as the fact that the real shocks are heterogenous (or "local") whereas the nominal shock is global, affecting the two countries in the same way. The interest of cooperation grows with the growing incidence of this heterogeneity. Formally, as the main interest of a federal scheme is to eliminate the incidence of the nominal shock, this interest vanishes relatively when the variance of the nominal shocks decreases relatively to the variance of the real shocks.

Let us develop a numerical example in order to better understand the choice between a federation and a cooperating scheme. Let us assume the following values for the parameters of the model:

$$\alpha=3,\beta=1,\gamma=1/2,\theta=1,b=1,c=1.$$

Then

$$E\left(L_{iF}^{*}\right)=\frac{1}{2}\frac{1\left(\frac{1}{2}+1\right)^{2}\left(1+(4)^{2}\right)}{1^{2}\left(\frac{7}{2}\right)^{2}}\chi^{2}+\frac{2\frac{1}{4}}{4}\sigma_{u}^{2}=\frac{153}{98}\chi^{2}+\frac{1}{8}\sigma_{u}^{2}$$

and

$$E\left(L_{iC}^{*}\right)=\frac{68}{100}\chi^{2}+\frac{1}{40}\sigma_{u}^{2}+\frac{1}{20}\sigma_{\varepsilon}^{2}.$$

Hence

$$E\left(L_{iF}^{*}\right)=\frac{153}{98}\chi^{2}+\frac{1}{8}\sigma_{u}^{2}\geq E\left(L_{iC}^{*}\right)=\frac{68}{100}\chi^{2}+\frac{1}{40}\sigma_{u}^{2}+\frac{1}{20}\sigma_{\varepsilon}^{2}$$

$$\Leftrightarrow\sigma_{u}^{2}\geq\frac{1}{2}\sigma_{\varepsilon}^{2}-66,64\chi^{2}.$$

Fiscal federalism is better than cooperation for both national fiscal authorities for the set of couples $\left(\sigma_u^2, \sigma_\varepsilon^2\right)$ satisfying this inequality. Of course, cooperation generates lower expected losses than the absence of cooperation. When a fiscal federation is preferable to cooperation, it is thus preferable to non-cooperation. But the inverse is not true. If cooperation is preferable, but impossible to implement, it does not imply that a federal scheme should be put in place. It may be that the expected losses obtained through non-cooperation is weaker than the one obtained with a federal scheme. The advantage of a supplementary instrument (thanks to a federal Treasury) may be lost because of the non-cooperative effects between three non-cooperating (and not two) players. From equations given the above table, we get:

$$E\left(L_{in}^*\right)^2 < E\left(L_{iF}^*\right)^2 \Leftrightarrow \sigma_u^2 > \frac{\left(A_{in}^* - A_{iF}^*\right)}{\left(B_{iF}^* - B_{in}^*\right)}\chi^2 + \frac{C_{in}^*}{\left(B_{iF}^* - B_{in}^*\right)}\sigma_\varepsilon^2. \tag{32}$$

Again, for given values of the structural parameters, non-cooperation is more enticing, the lower is the variance of the nominal shock as it decreases the interest of a federation.

5 Conclusion.

A monetary union between several countries reinforces the fiscal cross-border effects of fiscal decisions made by independent national fiscal authorities. It is thus necessary to ask about the best way to articulate these decisions. Given the absence of insulation from country shocks and fiscal decisions, a non-cooperating scheme is suboptimal and likely to amplify negative spillovers potentially damaging for the monetary union itself.

Two ways are possible to overcome the pitfalls of the absence of cooperation: the formation of a fiscal federation and intergovernmental cooperation. They imply different architectures and policy instruments. Each option has its merits and drawbacks.

As far as a fiscal federation is concerned, either economic analysis or evidence show that it cannot be a panacea. Its success depends on the type of fiscal discipline imposed on national fiscal authorities. Cooperation, on the other hand, is made fragile by the difficulties encountered in the bargaining process, including the building of a consensus on the macroeconomic outcome and because its implementation has to be renewed year after year. In brief we should suspect that no option is clearly better than the other.

The analysis of a simple macroeconomic model which can be developed alternatively in the configurations of fiscal federalism or of cooperation leads to an important conclusion: it is a priori impossible to prefer either one or the other of these options. It depends on the variances of shocks and the (disagreement between the) objectives of the various policymakers.

A corollary of this result is that a precise analysis of the economic structure of the monetary union as well as the policy objectives of the governments (or the electorates in a political economy perspective) is needed when choosing its institutional design. The choice of the adequate institutional design of fiscal policy making in a monetary union is a pragmatic matter, not a matter of principal.

This theoretical result sheds some light on the current travails of the European monetary union. We cannot be surprised to witness the hesitations in the building of a stronger union. They are clear evidence of a dilemma in choosing between federation and the option intergovernmental cooperation. For the time being, maybe temporarily, the supremacy of the European council in particular in dealing with the Greek debt drama proves that the scheme chosen by the European authorities is de facto the cooperation option. Our results show that it is too simple to a priori condemn this choice without further analysis.

The undetermined coefficients method we use to reach this result can be used to analyze more complex models than the one we study here. We chose a simple model in order to reach the agnostic result stated above. But it can be obtained in more complex models, including in particular a dynamic dimension missing here. This opens many complex issues not raised here, including time-consistency and trust. This is left for further research. Finally we abstract from the discussion on the relative difficulties of implementing these institutional design and then on the effectiveness of fiscal policy. These matters are crucial in the ranking of and the choice among institutional settings.

In sum, once it is recognized that insulation within a monetary union is impossible and its fiscal side plays a crucial role in its functioning and achievements, no simple view can be held on the proper fiscal institutions of a monetary union.

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Appendix.

From (4), (1) et (3), we get the following expression for output in country i:

$$y_i = \hat{y} + (\alpha + bc)\, g_i + (\beta + bc)\, g_{-i} + (\gamma + bc)\, g_F + b\,(\varepsilon - \pi^e) + u_i \tag{33}$$

Non-cooperation.

Here there is no Federal Treasury, the central bank has no active policy and applies $\pi_M = 0$ in any circumstance. Moreover we assume $\theta_M = \theta$ et $\chi_M \equiv \hat{y} - \tilde{\chi}_M = 0$.

Identification of reduced forms

The optimization program of Treasury i is (using (33)):

$$max_{g_i} L = \frac{1}{2}\theta\left((\alpha + bc)\, g_i + (\beta + bc)\, g_{-i} + b\,(\varepsilon - \pi^e) + u_i - \chi\right)^2 + \frac{1}{2}\left(c\,(g_1 + g_2) + \varepsilon - \pi^e\right)^2 \tag{34}$$

with $\chi = \tilde{\chi} - \hat{y}$.

The first-order condition is:

$$\frac{\partial L}{\partial g_i} = \theta\left((\alpha + bc)\, g_i + (\beta + bc)\, g_{-i} + b\,(\varepsilon - \pi^e) + u_i - \chi\right)(\alpha + bc) + c\,(c\,(g_1 + g_2) + \varepsilon) = 0$$

ou equivalently:

$$g_i = \frac{\theta\,(\alpha + bc)\,\chi - \left(\theta\,(\beta + bc)\,(\alpha + bc) + c^2\right) g_{-i} - \theta\,(\alpha + bc)\, u_i}{\left(\theta\,(\alpha + bc)^2 + c^2\right)}$$
$$- \frac{(c + \theta b\,(\alpha + bc))}{\left(\theta\,(\alpha + bc)^2 + c^2\right)}\varepsilon + \frac{\theta b\,(\alpha + bc)}{\left(\theta\,(\alpha + bc)^2 + c^2\right)}\pi^e.$$

To compute the equilibrium of the game, we use the undetermined coefficients method.[8] We denote g_{in}^{*} the optimal solutions of this game and π_n^e the corresponding expected inflation. We look for solutions depending on the shocks and the constants of the problem, that is the objectives of the players (here solely χ). We assume the following reduced forms:

$$g_{in}^{*} = f_{ni\chi}\chi + f_{nii}u_i + f_{nij}u_j + f_{ni\varepsilon}\varepsilon$$

$$\pi_n^e = f_{ne\chi}\chi.$$

The equilibrium solution is thus characterized by:

$$f_{ni\chi}\chi + f_{nii}u_i + f_{nij}u_j + f_{i\varepsilon}\varepsilon =$$

$$\frac{\theta\left(\alpha+bc\right)\chi - \left(\theta\left(\beta+bc\right)\left(\alpha+bc\right)+c^2\right)\left(f_{j\chi}\chi + f_{nji}u_i + f_{njj}u_j + f_{nj\varepsilon}\varepsilon\right)}{\left(\theta\left(\alpha+bc\right)^2+c^2\right)}$$

$$\frac{-\theta\left(\alpha+bc\right)u_i - \left(c+\theta b\left(\alpha+bc\right)\right)\varepsilon + \theta b\left(\alpha+bc\right)f_{ne\chi}\chi}{\left(\theta\left(\alpha+bc\right)^2+c^2\right)}$$

which implies:

$$f_{n2\chi} = \frac{\theta\left(\alpha+bc\right) - \left(\theta\left(\beta+bc\right)\left(\alpha+bc\right)+c^2\right)f_{n1\chi} + \theta b\left(\alpha+bc\right)f_{ne\chi}}{\left(\theta\left(\alpha+bc\right)^2+c^2\right)}$$

$$f_{n22} = \frac{-\left(\theta\left(\beta+bc\right)\left(\alpha+bc\right)+c^2\right)f_{n12} - \theta\left(\alpha+bc\right)}{\left(\theta\left(\alpha+bc\right)^2+c^2\right)}$$

$$f_{n21} = \frac{-\left(\theta\left(\beta+bc\right)\left(\alpha+bc\right)+c^2\right)f_{n11}}{\left(\theta\left(\alpha+bc\right)^2+c^2\right)}$$

$$f_{n2\varepsilon} = \frac{-\left(\theta\left(\beta+bc\right)\left(\alpha+bc\right)+c^2\right)f_{n1\varepsilon} - \left(c+\theta b\left(\alpha+bc\right)\right)}{\left(\theta\left(\alpha+bc\right)^2+c^2\right)}.$$

Since countries are identical, we can write:

$$f_{n1\chi} = f_{n2\chi}, f_{n11} = f_{n22}, f_{n1\varepsilon} = f_{n2\varepsilon}, f_{n12} = f_{n21}$$

As:

$$\pi = c\left(g_1+g_2\right)+\varepsilon,$$

we get:

$$f_{ne\chi} = c\left(f_{n1\chi}+f_{n2\chi}\right)$$

and thus:

$$f_{ne\chi} = 2cf_{n1\chi}$$

After tedious computation, we get:

$$f_{n1\chi} = f_{n2\chi} = \frac{\theta\left(\alpha+bc\right)}{\theta\left(\alpha+bc\right)\left(\alpha+\beta\right)+2c^2}$$

[8]This method used to be used to find the reduced forms of linear equations with rational expectations. To the best of our knowledge it is not be used to solve for the equilibrium of a set of interdependent equations with rational expectations.

$$f_{n1\varepsilon} = f_{n2\varepsilon} = -\frac{c+\theta b\left(\alpha+bc\right)}{\theta\left(\alpha+bc\right)\left(\alpha+\beta+2bc\right)+2c^{2}}$$

$$f_{n11} = f_{n22} = -\frac{\left(\theta\left(\alpha+bc\right)^{2}+c^{2}\right)}{\left[\theta\left(\alpha+bc\right)\left(\alpha+\beta+2bc\right)+2c^{2}\right]\left(\alpha-\beta\right)}$$

$$f_{n21} = f_{n12} = \frac{\left(\theta\left(\beta+bc\right)\left(\alpha+bc\right)+c^{2}\right)}{\left[\theta\left(\alpha+bc\right)\left(\alpha+\beta+2bc\right)+2c^{2}\right]\left(\alpha-\beta\right)}.$$

This ends the identification of the coefficients of the reduced forms. To summarize the reduced forms are given by the following relations:

$$g_{in}^{*} = \frac{\theta\left(\alpha+bc\right)}{\theta\left(\alpha+bc\right)\left(\alpha+\beta\right)+2c^{2}}\chi - \frac{\left(c+\theta b\left(\alpha+bc\right)\right)}{\theta\left(\alpha+bc\right)\left(\alpha+\beta+2bc\right)+2c^{2}}\varepsilon$$

$$-\frac{\left(\theta\left(\alpha+bc\right)^{2}+c^{2}\right)}{\left[\theta\left(\alpha+bc\right)\left(\alpha+\beta+2bc\right)+2c^{2}\right]\left(\alpha-\beta\right)}u_{i} + \frac{\left(\theta\left(\beta+bc\right)\left(\alpha+bc\right)+c^{2}\right)}{\left[\theta\left(\alpha+bc\right)\left(\alpha+\beta+2bc\right)+2c^{2}\right]\left(\alpha-\beta\right)}u_{j}.$$

For inflation we get:

$$\pi_{n}^{*} = c\left(g_{1}+g_{2}\right)+\varepsilon$$

$$= \frac{2c\theta\left(\alpha+bc\right)}{\theta\left(\alpha+bc\right)\left(\alpha+\beta\right)+2c^{2}}\chi + \frac{\theta\left(\alpha+bc\right)}{\theta\left(\alpha+bc\right)\left(\alpha+\beta+2bc\right)+2c^{2}}\left[-c\left(u_{1}+u_{2}\right)+\left(\alpha+\beta\right)\varepsilon\right].$$

The national aggregate outputs are equal to:

$$y_{in}^{*} = \hat{y} + \alpha g_{in}^{*} + \beta g_{jn}^{*} + b\left(\pi-\pi^{e}\right) + u_{1}$$

Hence using the expressions obtained above we get:

$$y_{in}^{*} = \hat{y} + \frac{\theta\left(\alpha+bc\right)\left(\alpha+\beta\right)}{\theta\left(\alpha+bc\right)\left(\alpha+\beta\right)+2c^{2}}\chi$$

$$+\frac{c^{2}}{\theta\left(\alpha+bc\right)\left(\alpha+\beta+2bc\right)+2c^{2}}\left(u_{1}+u_{2}\right) - \frac{c\left(\alpha+\beta\right)}{\theta\left(\alpha+bc\right)\left(\alpha+\beta+2bc\right)+2c^{2}}\varepsilon$$

The union's aggregate output is equal to:

$$\overline{y}_{n}^{*} = \hat{y} + \frac{\theta\left(\alpha+bc\right)\left(\alpha+\beta\right)}{\theta\left(\alpha+bc\right)\left(\alpha+\beta\right)+2c^{2}}\chi$$

$$+\frac{c^{2}}{\left[\theta\left(\alpha+bc\right)\left(\alpha+\beta+2bc\right)+2c^{2}\right]}\overline{u} - \frac{c\left(\alpha+\beta\right)}{\theta\left(\alpha+bc\right)\left(\alpha+\beta+2bc\right)+2c^{2}}\varepsilon$$

Expected losses.

Loss values are equal to:

- for the national fiscal authorities (taking into account that $\chi = \tilde{\chi} - \hat{y}$) :

$$L_{in}^{*}=\frac{1}{2}\theta\left\{\left(\frac{\theta\left(\alpha+bc\right)\left(\alpha+\beta\right)}{\theta\left(\alpha+bc\right)\left(\alpha+\beta\right)+2c^{2}}-1\right)\chi\right.$$
$$\left.+\frac{c^{2}}{\theta\left(\alpha+bc\right)\left(\alpha+\beta+2bc\right)+2c^{2}}\left(u_{1}+u_{2}\right)-\frac{c\left(\alpha+\beta\right)}{\theta\left(\alpha+bc\right)\left(\alpha+\beta+2bc\right)+2c^{2}}\varepsilon\right\}^{2}$$
$$+\frac{1}{2}\left\{\frac{2c\theta\left(\alpha+bc\right)}{\theta\left(\alpha+bc\right)\left(\alpha+\beta\right)+2c^{2}}\chi+\frac{\theta\left(\alpha+bc\right)}{\theta\left(\alpha+bc\right)\left(\alpha+\beta+2bc\right)+2c^{2}}\left[\left(\alpha+\beta\right)\varepsilon-c\left(u_{1}+u_{2}\right)\right]\right\}^{2}.$$

- for the central bank (taking into account that $\chi_M \equiv \hat{y} - \tilde{\chi}_M = 0$):

$$L_{Mn}^{*}=\frac{1}{2}\theta_{M}\left\{\left(\frac{\theta\left(\alpha+bc\right)\left(\alpha+\beta\right)}{\theta\left(\alpha+bc\right)\left(\alpha+\beta\right)+2c^{2}}\chi\right)\right.$$
$$\left.+\frac{c^{2}}{\theta\left(\alpha+bc\right)\left(\alpha+\beta+2bc\right)+2c^{2}}\left(u_{1}+u_{2}\right)-\frac{c\left(\alpha+\beta\right)}{\theta\left(\alpha+bc\right)\left(\alpha+\beta+2bc\right)+2c^{2}}\varepsilon\right\}^{2}$$
$$+\frac{1}{2}\left\{\frac{2c\theta\left(\alpha+bc\right)}{\theta\left(\alpha+bc\right)\left(\alpha+\beta\right)+2c^{2}}\chi+\frac{\theta\left(\alpha+bc\right)}{\theta\left(\alpha+bc\right)\left(\alpha+\beta+2bc\right)+2c^{2}}\left[\left(\alpha+\beta\right)\varepsilon-c\left(u_{1}+u_{2}\right)\right]\right\}^{2}.$$

Expected losses are thus equal to:

- for the national fiscal authorities:

$$E\left(L_{in}^{*}\right)^{2}=\frac{2\theta c^{2}\left(c^{2}+\theta\left(\alpha+bc\right)^{2}\right)}{\left(\theta\left(\alpha+bc\right)\left(\alpha+\beta\right)+2c^{2}\right)^{2}}\chi^{2}$$
$$+\frac{\theta c^{2}\left(c^{2}+\theta\left(\alpha+bc\right)^{2}\right)}{\left(\theta\left(\alpha+bc\right)\left(\alpha+\beta+2bc\right)+2c^{2}\right)^{2}}\sigma_{u}^{2}+\frac{1}{2}\frac{\theta c^{2}\left(\left(\alpha+\beta\right)^{2}+\theta\left(\alpha+bc\right)^{2}\right)}{\left(\theta\left(\alpha+bc\right)\left(\alpha+\beta+2bc\right)+2c^{2}\right)^{2}}\sigma_{\varepsilon}^{2}.$$

- for the central bank:

$$E\left(L_{Mn}^{*}\right)^{2}=\frac{1}{2}\frac{\theta^{2}\left(\alpha+bc\right)^{2}\left(\theta_{M}\left(\alpha+\beta\right)+4c^{2}\right)}{\left(\theta\left(\alpha+bc\right)\left(\alpha+\beta\right)+2c^{2}\right)^{2}}\chi^{2}$$
$$+\frac{c^{2}\left(\theta_{M}c^{2}+\theta^{2}\left(\alpha+bc\right)^{2}\right)}{\left(\theta\left(\alpha+bc\right)\left(\alpha+\beta+2bc\right)+2c^{2}\right)^{2}}\sigma_{u}^{2}+\frac{1}{2}\frac{\left(\alpha+\beta\right)^{2}\left(\theta_{M}c^{2}+\theta^{2}\left(\alpha+bc\right)^{2}\right)}{\left(\theta\left(\alpha+bc\right)\left(\alpha+\beta+2bc\right)+2c^{2}\right)^{2}}\sigma_{\varepsilon}^{2}.$$

Fiscal cooperation.

In this scenario we assume no federal Treasury, no active monetary policy $\pi_M = 0$, and cooperation between national fiscal authorities.

Identification of reduced forms The program of the cooperating national fiscal authorities is:

$$max_{g_{1},g_{2}}L=\frac{1}{2}\frac{\theta}{2}\left(\left(\alpha+bc\right)g_{1}+\left(\beta+bc\right)g_{2}+b\left(\varepsilon-\pi^{e}\right)+u_{1}-\chi\right)^{2} \tag{35}$$
$$+\frac{1}{2}\frac{\theta}{2}\left(\left(\alpha+bc\right)g_{2}+\left(\beta+bc\right)g_{1}+b\left(\varepsilon-\pi^{e}\right)+u_{2}-\chi\right)^{2}+\frac{1}{2}\left(c\left(g_{1}+g_{2}\right)+\varepsilon\right)^{2}$$

with $\chi \equiv \tilde{\chi} - \hat{y}$.

The first-order condition for g_i is:

$$\frac{\partial L}{\partial g_i} = \frac{\theta}{2}\left(\left(\alpha+bc\right)g_i+\left(\beta+bc\right)g_j+b\left(\varepsilon-\pi^e\right)+u_i-\chi\right)\left(\alpha+bc\right)$$
$$+\frac{\theta}{2}\left(\left(\alpha+bc\right)g_j+\left(\beta+bc\right)g_i+b\left(\varepsilon-\pi^e\right)+u_j-\chi\right)\left(\beta+bc\right)$$
$$+c\left(c\left(g_1+g_2\right)+\varepsilon\right)=0$$

or equivalently :

$$g_i = \frac{\frac{\theta}{2}b\left(\alpha+\beta+2bc\right)}{\left[\frac{\theta}{2}\left(\left(\alpha+bc\right)^2+\left(\beta+bc\right)^2\right)+c^2\right]}\pi^e+\frac{\frac{\theta}{2}\left(\alpha+\beta+2bc\right)}{\left[\frac{\theta}{2}\left(\left(\alpha+bc\right)^2+\left(\beta+bc\right)^2\right)+c^2\right]}\chi$$
$$-\frac{\left[\theta\left(\alpha+bc\right)\left(\beta+bc\right)+c^2\right]}{\left[\frac{\theta}{2}\left(\left(\alpha+bc\right)^2+\left(\beta+bc\right)^2\right)+c^2\right]}g_j$$
$$-\frac{\left[c+\frac{\theta}{2}b\left(\alpha+\beta+2bc\right)\right]}{\left[\frac{\theta}{2}\left(\left(\alpha+bc\right)^2+\left(\beta+bc\right)^2\right)+c^2\right]}\varepsilon-\frac{\frac{\theta}{2}\left(\left(\alpha+bc\right)u_i+\left(\beta+bc\right)u_j\right)}{\left[\frac{\theta}{2}\left(\left(\alpha+bc\right)^2+\left(\beta+bc\right)^2\right)+c^2\right]}.$$

We denote by g_{iC}^* the optimal solutions of this game and π_C^e the corresponding expected inflation. The reduced forms for these solutions are written as follows:

$$g_{iC}^* = f_{Ci\chi}\chi+f_{Cii}u_i+f_{Cij}u_j+f_{Ci\varepsilon}\varepsilon$$

$$\pi_C^e = f_{Ce}\chi.$$

We then get:

$$f_{Ci\chi}\chi+f_{Cii}u_i+f_{Cij}u_j+f_{Ci\varepsilon}\varepsilon =$$
$$\frac{\frac{\theta}{2}b\left(\alpha+\beta+2bc\right)f_{Ce}+\frac{\theta}{2}\left(\alpha+\beta+2bc\right)-\left[\theta\left(\alpha+bc\right)\left(\beta+bc\right)+c^2\right]f_{Cj\chi}}{\left[\frac{\theta}{2}\left(\left(\alpha+bc\right)^2+\left(\beta+bc\right)^2\right)+c^2\right]}\chi$$
$$-\frac{\left[\theta\left(\alpha+bc\right)\left(\beta+bc\right)+c^2\right]f_{Cji}+\frac{\theta}{2}\left(\alpha+bc\right)}{\left[\frac{\theta}{2}\left(\left(\alpha+bc\right)^2+\left(\beta+bc\right)^2\right)+c^2\right]}u_i-\frac{\left[\theta\left(\alpha+bc\right)\left(\beta+bc\right)+c^2\right]f_{Cjj}+\frac{\theta}{2}\left(\left(\beta+bc\right)\right)}{\left[\frac{\theta}{2}\left(\left(\alpha+bc\right)^2+\left(\beta+bc\right)^2\right)+c^2\right]}u_j$$
$$-\frac{\left[c+\frac{\theta}{2}b\left(\alpha+\beta+2bc\right)\right]+\left[\theta\left(\alpha+bc\right)\left(\beta+bc\right)+c^2\right]f_{Cj\varepsilon}}{\left[\frac{\theta}{2}\left(\left(\alpha+bc\right)^2+\left(\beta+bc\right)^2\right)+c^2\right]}\varepsilon.$$

At the symmetrical equilibrium we get:

$$f_{C1\chi}=f_{C2\chi}, f_{C11}=f_{C22}, f_{C1\varepsilon}=f_{C2\varepsilon}, f_{C12}=f_{C21}.$$

Moreover:

$$f_{Ce}=c\left(f_{C1\chi}+f_{C2\chi}\right).$$

Hence:

$$f_{Ce}=2cf_{C1\chi}.$$

Combining these equations we get:

$$f_{Ci\chi}=\frac{\frac{\theta}{2}\left(\alpha+\beta+2bc\right)}{\left[\frac{\theta}{2}\left(\alpha+\beta+2bc\right)\left(\alpha+\beta\right)+2c^2\right]}$$

$$f_{Ci\varepsilon} = -\frac{\left[c + \frac{\theta}{2} b \left(\alpha + \beta + 2bc\right)\right]}{\left[\frac{\theta}{2} \left(\alpha + \beta + 2bc\right)^2 + 2c^2\right]}$$

$$f_{Cii} = -\frac{\frac{\theta}{2} \left(\alpha + \beta + 2bc\right) \left(\alpha + bc\right) + c^2}{\left(\alpha - \beta\right) \left[\frac{\theta}{2} \left(\alpha + \beta + 2bc\right)^2 + 2c^2\right]}$$

$$f_{Cij} = f_{Cji} = \frac{\frac{\theta}{2} \left(\alpha + \beta + 2bc\right) \left(\beta + bc\right) + c^2}{\left(\alpha - \beta\right) \left[\frac{\theta}{2} \left(\alpha + \beta + 2bc\right)^2 + 2c^2\right]}.$$

Replacing these expressions in the reduced forms we get:

$$g^*_{iC} = \frac{\frac{\theta}{2} \left(\alpha + \beta + 2bc\right)}{\left[\frac{\theta}{2} \left(\alpha + \beta + 2bc\right) \left(\alpha + \beta\right) + 2c^2\right]} \chi - \frac{\left[c + \frac{\theta}{2} b \left(\alpha + \beta + 2bc\right)\right]}{\left[\frac{\theta}{2} \left(\alpha + \beta + 2bc\right)^2 + 2c^2\right]} \varepsilon$$

$$-\frac{\frac{\theta}{2} \left(\alpha + \beta + 2bc\right) \left(\alpha + bc\right) + c^2}{\left(\alpha - \beta\right) \left[\frac{\theta}{2} \left(\alpha + \beta + 2bc\right)^2 + 2c^2\right]} u_i + \frac{\frac{\theta}{2} \left(\alpha + \beta + 2bc\right) \left(\beta + bc\right) + c^2}{\left(\alpha - \beta\right) \left[\frac{\theta}{2} \left(\alpha + \beta + 2bc\right)^2 + 2c^2\right]} u_j.$$

We also obtain the following expression for inflation which we denote π^*_C:

$$\pi^*_C = -\frac{\frac{\theta}{2} c \left(\alpha + \beta + 2bc\right)}{\frac{\theta}{2} \left(\alpha + \beta + 2bc\right)^2 + 2c^2} \left(u_1 + u_2\right) + \frac{\left[\frac{\theta}{2} \left(\alpha + \beta + 2bc\right)\right] \left(\alpha + \beta\right)}{\left[\frac{\theta}{2} \left(\alpha + \beta + 2bc\right)^2 + 2c^2\right]} \varepsilon.$$

Using this expression, the national aggregate output are equal to:

$$y^*_{1C} = y^*_{2C} = \hat{y} + \left(\alpha g^*_{1C} + \beta g^*_{2C}\right) + b \left(\pi - \pi^e\right) + u_1$$

$$= \hat{y} + \frac{\theta c \left(\alpha + \beta + 2bc\right) \left(\alpha + \beta\right)}{\left[\frac{\theta}{2} \left(\alpha + \beta + 2bc\right) \left(\alpha + \beta\right) + 2c^2\right]} \chi + \frac{c^2 \left(u_1 + u_2\right) - c \left(\alpha + \beta\right) \varepsilon}{\left[\frac{\theta}{2} \left(\alpha + \beta + 2bc\right)^2 + 2c^2\right]}.$$

We check $y^*_{1C} = y^*_{2C}$, which is consistent with cooperation, given the symmetrical structures of the two countries.

Expected losses.

Actual losses are equal to (taking into account that $\chi \equiv \tilde{\chi} - \hat{y}$ and $\chi_M \equiv \hat{y} - \tilde{\chi}_M = 0$) :

- for national fiscal authorities:

$$L^*_{iC} = \frac{1}{2} \theta \left(y^*_{iC} - \tilde{\chi}\right)^2 + \frac{1}{2} \left(\pi\right)^2$$

$$= \frac{1}{2} \theta \left(\frac{\frac{\theta}{2} \left(\alpha + \beta + 2bc\right) \left(\alpha + \beta\right) - 2c^2}{\frac{\theta}{2} \left(\alpha + \beta + 2bc\right) \left(\alpha + \beta\right) + 2c^2} \chi + \frac{c^2 \left(u_1 + u_2\right) - c \left(\alpha + \beta\right) \varepsilon}{\left[\frac{\theta}{2} \left(\alpha + \beta + 2bc\right)^2 + 2c^2\right]} \right)^2$$

$$+ \frac{1}{2} \left(\frac{\theta c \left(\alpha + \beta + 2bc\right)}{\left[\frac{\theta}{2} \left(\alpha + \beta + 2bc\right) \left(\alpha + \beta\right) + 2c^2\right]} \chi + \frac{\frac{\theta}{2} \left(\alpha + \beta + 2bc\right) \left(\left(\alpha + \beta\right) \varepsilon - c \left(u_1 + u_2\right)\right)}{\frac{\theta}{2} \left(\alpha + \beta + 2bc\right)^2 + 2c^2} \right)^2$$

and thus the expected loss is equal to:

$$E\left(L_{iC}^{*}\right)=\frac{1}{2}\theta\frac{\left(\frac{\theta}{2}\left(\alpha+\beta+2bc\right)\left(\alpha+\beta\right)-2c^{2}\right)^{2}+\theta c^{2}\left(\alpha+\beta+2bc\right)^{2}}{\left(\frac{\theta}{2}\left(\alpha+\beta+2bc\right)\left(\alpha+\beta\right)+2c^{2}\right)^{2}}\chi^{2}$$

$$+\frac{c^{2}\left(\left(\frac{\theta}{2}\right)^{2}\left(\alpha+\beta+2bc\right)^{2}+\theta c^{2}\right)}{\left(\frac{\theta}{2}\left(\alpha+\beta+2bc\right)^{2}+2c^{2}\right)^{2}}\sigma_{u}^{2}+\frac{1}{2}\frac{\left(\left(\frac{\theta}{2}\right)^{2}\left(\alpha+\beta+2bc\right)^{2}+\theta c^{2}\right)\left(\alpha+\beta\right)}{\left(\frac{\theta}{2}\left(\alpha+\beta+2bc\right)^{2}+2c^{2}\right)^{2}}\sigma_{\varepsilon}^{2}.$$

- for the central bank:

$$L_{MC}^{*}=\frac{1}{2}\theta_{M}\left(\overline{y}_{C}^{*}-\tilde{\chi}_{M}\right)^{2}+\frac{1}{2}\pi^{2}$$

$$=\frac{1}{2}\theta_{M}\left(\frac{\theta c\left(\alpha+\beta+2bc\right)\left(\alpha+\beta\right)}{\left[\frac{\theta}{2}\left(\alpha+\beta+2bc\right)\left(\alpha+\beta\right)+2c^{2}\right]}\chi+\frac{c^{2}\left(u_{1}+u_{2}\right)-c\left(\alpha+\beta\right)\varepsilon}{\left[\frac{\theta}{2}\left(\alpha+\beta+2bc\right)^{2}+2c^{2}\right]}\right)^{2}$$

$$+\frac{1}{2}\left(\frac{\theta c\left(\alpha+\beta+2bc\right)}{\left[\frac{\theta}{2}\left(\alpha+\beta+2bc\right)\left(\alpha+\beta\right)+2c^{2}\right]}\chi+\frac{\frac{\theta}{2}\left(\alpha+\beta+2bc\right)\left(\left(\alpha+\beta\right)\varepsilon-c\left(u_{1}+u_{2}\right)\right)}{\frac{\theta}{2}\left(\alpha+\beta+2bc\right)^{2}+2c^{2}}\right)^{2}$$

and thus the expected loss is equal to:

$$E\left(L_{MC}^{*}\right)=\frac{1}{2}\frac{\theta^{2}c^{2}\left(\alpha+\beta+2bc\right)^{2}}{\left[\frac{\theta}{2}\left(\alpha+\beta+2bc\right)\left(\alpha+\beta\right)+2c^{2}\right]^{2}}\left(\theta_{M}\left(\alpha+\beta\right)^{2}+1\right)\chi^{2}$$

$$+\frac{1}{2}\frac{\left(\alpha+\beta\right)^{2}c^{2}\left(\theta_{M}c^{2}+\left(\frac{\theta}{2}\right)^{2}\left(\alpha+\beta+2bc\right)^{2}\right)}{\left[\frac{\theta}{2}\left(\alpha+\beta+2bc\right)^{2}+2c^{2}\right]^{2}}\sigma_{\varepsilon}^{2}+\frac{c^{2}\left(\theta_{M}c^{2}+\left(\frac{\theta}{2}\right)^{2}\left(\alpha+\beta+2bc\right)^{2}\right)}{\left[\frac{\theta}{2}\left(\alpha+\beta+2bc\right)^{2}+2c^{2}\right]^{2}}\sigma_{u}^{2}.$$

Fiscal federalism.

In this scenario we assume the existence of federal Treasury, no active monetary policy $\pi_M = 0$, and no cooperation between national fiscal authorities. We assume that $\theta_F = \theta$ and $\chi_F = 0$.

Reduced form identification.

The optimization program of the national fiscal authority i is:

$$max_{g_i}L=\frac{1}{2}\theta\left(\left(\alpha+bc\right)g_{i}+\left(\beta+bc\right)g_{-i}+\left(\gamma+bc\right)g_{F}+b\left(\varepsilon-\pi^{e}\right)+u_{i}-\chi\right)^{2}+\frac{1}{2}\left(c\left(g_{1}+g_{2}+g_{F}\right)+\varepsilon\right)^{2} \tag{36}$$

with $\chi = \tilde{\chi} - \hat{y}$.

The first-order condition is:

$$\frac{\partial L}{\partial g_{i}}=\theta\left(\left(\alpha+bc\right)g_{i}+\left(\beta+bc\right)g_{-i}+\left(\gamma+bc\right)g_{F}+b\left(\varepsilon-\pi^{e}\right)+u_{i}-\chi\right)\left(\alpha+bc\right)+c\left(c\left(g_{1}+g_{2}+g_{F}\right)+\varepsilon\right)$$

$$+c\left(c\left(g_{1}+g_{2}+g_{F}\right)+\varepsilon\right)=0$$

or equivalently:

$$g_i = \frac{\theta(\alpha+bc)\chi - \theta(\alpha+bc)u_i - (c+\theta b(\alpha+bc))\varepsilon + \theta b(\alpha+bc)\pi^e}{\left(\theta(\alpha+bc)^2+c^2\right)}$$

$$-\frac{\left(\theta(\beta+bc)(\alpha+bc)+c^2\right)}{\left(\theta(\alpha+bc)^2+c^2\right)}g_{-i} - \frac{\left(\theta(\gamma+bc)(\alpha+bc)+c^2\right)}{\left(\theta(\alpha+bc)^2+c^2\right)}g_F.$$

The optimization program of the federal Treasury is:

$$max_{g_F} L_F = \frac{1}{2}\theta\left(\left(\frac{1}{2}(\alpha+\beta)+bc\right)(g_1+g_2)+(\gamma+bc)g_F+b(\varepsilon-\pi^e)+\bar{u}\right)^2$$

$$+\frac{1}{2}\left(c(g_1+g_2+g_F)+\varepsilon\right)^2.$$

The first-order condition is:

$$\frac{\partial L}{\partial g_F} = \theta(\gamma+bc)\left(\left(\frac{1}{2}(\alpha+\beta)+bc\right)(g_1+g_2)+(\gamma+bc)g_F+b(\varepsilon-\pi^e)+\bar{u}\right)$$

$$+c\left(c(g_1+g_2+g_F)+\varepsilon\right)=0$$

or equivalently:

$$g_F = -\frac{\left(\theta(\gamma+bc)\left(\frac{1}{2}(\alpha+\beta)+bc\right)+c^2\right)(g_1+g_2)}{\left(\theta(\gamma+bc)^2+c^2\right)}$$

$$-\frac{1}{2}\frac{\theta(\gamma+bc)}{\left(\theta(\gamma+bc)^2+c^2\right)}(u_1+u_2) - \frac{(b\theta(\gamma+bc)+c)}{\left(\theta(\gamma+bc)^2+c^2\right)}\varepsilon + \frac{b\theta(\gamma+bc)}{\left(\theta(\gamma+bc)^2+c^2\right)}\pi^e.$$

Denote by g^*_{Fi} et g^*_{FF} the optimal solutions of this program and π^e_F the corresponding expected inflation. The reduced forms for these solutions are written as follows:

$$g^*_{iF} = f_{Fi\chi}\chi + f_{Fii}u_i + f_{Fij}u_j + f_{Fi\varepsilon}\varepsilon$$

$$g^*_{FF} = f_{FF\chi}\chi + f_{FF1}u_1 + f_{FF2}u_2 + f_{FF\varepsilon}\varepsilon$$

$$\pi^e_F = f_{Fe\chi}\chi.$$

The equilibrium is characterized by:

$$f_{Fi\chi}\chi + f_{Fii}u_i + f_{Fij}u_j + f_{Fi\varepsilon}\varepsilon =$$

$$\frac{\theta(\alpha+bc)\chi - \theta(\alpha+bc)u_1 - (c+\theta b(\alpha+bc))\varepsilon}{\left(\theta(\alpha+bc)^2+c^2\right)} + \frac{\theta b(\alpha+bc)}{\left(\theta(\alpha+bc)^2+c^2\right)}f_{Fe\chi}\chi$$

$$-\frac{\left(\theta(\beta+bc)(\alpha+bc)+c^2\right)}{\left(\theta(\alpha+bc)^2+c^2\right)}\left(f_{Fj\chi}\chi + f_{Fji}u_i + f_{Fjj}u_j + f_{Fj\varepsilon}\varepsilon\right)$$

$$-\frac{\left(\theta(\gamma+bc)(\alpha+bc)+c^2\right)}{\left(\theta(\alpha+bc)^2+c^2\right)}\left(f_{FF\chi}\chi + f_{FFi}u_i + f_{FFj}u_j + f_{FF\varepsilon}\varepsilon\right).$$

Hence:

$$f_{Fi\chi}=\frac{\theta\left(\alpha+bc\right)}{\left(\theta\left(\alpha+bc\right)^{2}+c^{2}\right)}+\frac{b\theta\left(\alpha+bc\right)}{\left(\theta\left(\alpha+bc\right)^{2}+c^{2}\right)}f_{Fe\chi}$$

$$-\frac{\left(\theta\left(\beta+bc\right)\left(\alpha+bc\right)+c^{2}\right)}{\left(\theta\left(\alpha+bc\right)^{2}+c^{2}\right)}f_{Fj\chi}-\frac{\left(\theta\left(\gamma+bc\right)\left(\alpha+bc\right)+c^{2}\right)}{\left(\theta\left(\alpha+bc\right)^{2}+c^{2}\right)}f_{FF\chi}$$

$$f_{Fii}=\frac{-\theta\left(\alpha+bc\right)}{\left(\theta\left(\alpha+bc\right)^{2}+c^{2}\right)}$$

$$-\frac{\left(\theta\left(\beta+bc\right)\left(\alpha+bc\right)+c^{2}\right)}{\left(\theta\left(\alpha+bc\right)^{2}+c^{2}\right)}f_{Fji}-\frac{\left(\theta\left(\gamma+bc\right)\left(\alpha+bc\right)+c^{2}\right)}{\left(\theta\left(\alpha+bc\right)^{2}+c^{2}\right)}f_{FFi}$$

$$f_{Fij}=-\frac{\left(\theta\left(\beta+bc\right)\left(\alpha+bc\right)+c^{2}\right)}{\left(\theta\left(\alpha+bc\right)^{2}+c^{2}\right)}f_{Fjj}-\frac{\left(\theta\left(\gamma+bc\right)\left(\alpha+bc\right)+c^{2}\right)}{\left(\theta\left(\alpha+bc\right)^{2}+c^{2}\right)}f_{FFj}$$

$$f_{Fi\varepsilon}=-\frac{\left(c+\theta b\left(\alpha+bc\right)\right)}{\left(\theta\left(\alpha+bc\right)^{2}+c^{2}\right)}-\frac{\left(\theta\left(\beta+bc\right)\left(\alpha+bc\right)+c^{2}\right)}{\left(\theta\left(\alpha+bc\right)^{2}+c^{2}\right)}f_{Fj\varepsilon}$$

$$-\frac{\left(\theta\left(\gamma+bc\right)\left(\alpha+bc\right)+c^{2}\right)}{\left(\theta\left(\alpha+bc\right)^{2}+c^{2}\right)}f_{FF\varepsilon}.$$

We also get:

$$f_{FF\chi}\chi+f_{FF1}u_1+f_{FF2}u_2+f_{FF\varepsilon}\varepsilon=$$

$$-\frac{\left(\theta\left(\gamma+bc\right)\left(\frac{1}{2}\left(\alpha+\beta\right)+bc\right)+c^{2}\right)\left(f_{F1\chi}\chi+f_{F11}u_1+f_{F12}u_2+f_{F1\varepsilon}\varepsilon\right)}{\left(\theta\left(\gamma+bc\right)^{2}+c^{2}\right)}$$

$$-\frac{\theta\left(\gamma+bc\right)\left(\frac{1}{2}\left(\alpha+\beta\right)+bc+c^{2}\right)\left(f_{F2\chi}\chi+f_{F21}u_1+f_{F22}u_2+f_{F2\varepsilon}\varepsilon\right)}{\left(\theta\left(\gamma+bc\right)^{2}+c^{2}\right)}$$

$$-\frac{1}{2}\frac{\theta\left(\gamma+bc\right)}{\left(\theta\left(\gamma+bc\right)^{2}+c^{2}\right)}\left(u_1+u_2\right)-\frac{\left(b\theta\left(\gamma+bc\right)+c\right)}{\left(\theta\left(\gamma+bc\right)^{2}+c^{2}\right)}\varepsilon+\frac{b\theta\left(\gamma+bc\right)}{\left(\theta\left(\gamma+bc\right)^{2}+c^{2}\right)}f_{Fe\chi}\chi.$$

Hence:

$$f_{FF\chi}=-\frac{\left(\theta\left(\gamma+bc\right)\left(\frac{1}{2}\left(\alpha+\beta\right)+bc\right)+c^{2}\right)}{\left(\theta\left(\gamma+bc\right)^{2}+c^{2}\right)}\left(f_{F1\chi}+f_{F2\chi}\right)+\frac{b\theta\left(\gamma+bc\right)}{\left(\theta\left(\gamma+bc\right)^{2}+c^{2}\right)}f_{Fe\chi}$$

$$f_{FFi}=-\frac{\left(\theta\left(\gamma+bc\right)\left(\frac{1}{2}\left(\alpha+\beta\right)+bc\right)+c^{2}\right)\left(f_{Fii}+f_{Fji}\right)+\frac{1}{2}\theta\left(\gamma+bc\right)}{\left(\theta\left(\gamma+bc\right)^{2}+c^{2}\right)}$$

$$f_{FF\varepsilon}=-\frac{\left(\theta\left(\gamma+bc\right)\left(\frac{1}{2}\left(\alpha+\beta\right)+bc\right)+c^{2}\right)\left(f_{F1\varepsilon}+f_{F2\varepsilon}\right)+b\theta\left(\gamma+bc\right)+c}{\left(\theta\left(\gamma+bc\right)^{2}+c^{2}\right)}$$

$$f_{Fe\chi}=c\left(f_{F1\chi}+f_{F2\chi}+f_{FF\chi}\right).$$

Given the symmetry between the two countries, we can write:

$$f_{F1\chi}=f_{F2\chi},f_{F11}=f_{F22},f_{F1\varepsilon}=f_{F2\varepsilon},f_{F12}=f_{F21},f_{FF1}=f_{FF2}$$

Combining these equations we get:

$$f_{F1\chi} = f_{F2\chi} = \frac{(\alpha + bc)\left(\theta\gamma(\gamma + bc) + c^2\right)}{c^2(\gamma - \alpha)(2\gamma - \alpha - \beta)}$$

$$f_{FF\chi} = -\frac{\left(2c^2 + \theta(\gamma + bc)(\alpha + \beta)\right)(\alpha + bc)}{c^2(\gamma - \alpha)(2\gamma - \alpha - \beta)}$$

$$f_{FF\varepsilon} = \frac{(\alpha + \beta)}{c\,[2\gamma - \alpha - \beta]}$$

$$f_{F11} = f_{F22} = \frac{1}{2\gamma - \alpha - \beta}\frac{[\alpha - \gamma]}{[\alpha - \beta]}$$

$$f_{F12} = f_{F21} = \frac{(\gamma - \beta)}{(2\gamma - \alpha - \beta)(\alpha - \beta)}$$

$$f_{FF1} = f_{FF2} = -\frac{1}{2\gamma - \alpha - \beta}.$$

This completes the identification of the coefficients of the reduced forms.

The equilibrium solutions obtain:

$$g^*_{iF} = \frac{(\alpha + bc)\left(\theta\gamma(\gamma + bc) + c^2\right)}{c^2(\gamma - \alpha)(2\gamma - \alpha - \beta)}\chi - \frac{\gamma}{c\,[2\gamma - \alpha - \beta]}\varepsilon$$
$$+\frac{(\alpha - \gamma)}{(2\gamma - \alpha - \beta)(\alpha - \beta)}u_i + \frac{(\gamma - \beta)}{(\alpha - \beta)(2\gamma - \alpha - \beta)}u_j$$

$$g^*_{FF} = -\frac{\left(2c^2 + \theta(\gamma + bc)(\alpha + \beta)\right)(\alpha + bc)}{c^2(\gamma - \alpha)(2\gamma - \alpha - \beta)}\chi + \frac{(\alpha + \beta)}{c(2\gamma - \alpha - \beta)}\varepsilon$$
$$-\frac{1}{2\gamma - \alpha - \beta}u_1 - \frac{1}{2\gamma - \alpha - \beta}u_2.$$

For inflation we get:

$$\pi^*_F = c\left(g^*_{F1} + g^*_{F2} + g^*_{FF}\right) + \varepsilon = \frac{\theta(\gamma + bc)(\alpha + bc)}{c(\gamma - \alpha)}\chi.$$

The country i's aggregate product is equal to:

$$y^*_{iF} = \hat{y} + \left(\alpha g^*_{Fi} + \beta g^*_{F2} + \gamma g^*_{FF}\right) + b\left(\pi^*_F - (\pi^*_F)^e\right) + u_i$$
$$= \hat{y} - \frac{(\alpha + bc)}{(\gamma - \alpha)}\chi - \frac{2\gamma\beta}{(2\gamma - \alpha - \beta)(\alpha - \beta)}u_i.$$

Lastly, the union's aggregate output is equal to:

$$\overline{y}^*_F = \hat{y} - \frac{(\alpha + bc)}{(\gamma - \alpha)}\chi - \frac{2\gamma\beta}{(2\gamma - \alpha - \beta)(\alpha - \beta)}\overline{u}.$$

Expected losses.

We can derive the levels of loss for the various players (taking into account that $\chi \equiv \tilde{\chi} - \hat{y}$, $\chi_M \equiv \hat{y} - \tilde{\chi}_M = 0$ and $\chi_F \equiv \hat{y} - \tilde{\chi}_F = 0$) :

- for the national fiscal authority in i, the loss is equal to:

$$L_{iF}^{*} = \frac{1}{2}\theta\left(y_{iF}^{*} - \tilde{\chi}\right)^{2} + \frac{1}{2}\pi^{2}$$

$$= -\frac{1}{2}\theta\left(\frac{\gamma + bc}{(\gamma - \alpha)}\chi + \frac{2\gamma\beta}{(2\gamma - \alpha - \beta)(\alpha - \beta)}u_{i}\right)^{2} + \frac{1}{2}\left(\frac{\theta(\gamma + bc)(\alpha + bc)}{c(\gamma - \alpha)}\right)^{2}\chi^{2}$$

and thus the expected loss for the national fiscal authority in i is:

$$E\left(L_{iF}^{*}\right) = \frac{1}{2}\frac{\theta(\gamma + bc)^{2}\left(c^{2} + \theta(\alpha + bc)^{2}\right)}{c^{2}(\gamma - \alpha)^{2}}\chi^{2} + \theta\frac{2\gamma^{2}\beta^{2}}{(2\gamma - \alpha - \beta)^{2}(\alpha - \beta)^{2}}\sigma_{u}^{2}.$$

- for the central bank, the actual loss is:

$$L_{MF}^{*} = \frac{1}{2}\theta_{M}\left(\overline{y}_{F}^{*} - \tilde{\chi}_{M}\right)^{2} + \frac{1}{2}\pi^{2}$$

$$= -\frac{1}{2}\theta_{M}\left(\frac{(\alpha + bc)}{(\gamma - \alpha)}\chi + \frac{2\gamma\beta}{(2\gamma - \alpha - \beta)(\alpha - \beta)}\overline{u}\right)^{2} + \frac{1}{2}\left(\frac{\theta(\gamma + bc)(\alpha + bc)}{c(\gamma - \alpha)}\right)^{2}\chi^{2}$$

and thus the expected loss of central bank is:

$$E\left(L_{MF}^{*}\right) = \frac{1}{2}\frac{(\alpha + bc)^{2}\left(\theta_{M}c^{2} + \theta^{2}(\gamma + bc)^{2}\right)}{c^{2}(\gamma - \alpha)^{2}}\chi^{2} + \frac{2\theta_{M}\gamma^{2}\beta^{2}}{(2\gamma - \alpha - \beta)^{2}(\alpha - \beta)^{2}}\sigma_{u}^{2}.$$

- for the federal fiscal authority:

$$L_{FF}^{*} = \frac{1}{2}\theta_{F}\left(\overline{y}_{F}^{*} - \tilde{\chi}_{F}\right)^{2} + \frac{1}{2}\pi^{2}$$

$$= -\frac{1}{2}\theta_{F}\left(\frac{(\alpha + bc)}{(\gamma - \alpha)}\chi + \frac{2\gamma\beta}{(2\gamma - \alpha - \beta)(\alpha - \beta)}\overline{u}\right)^{2} + \frac{1}{2}\left(\frac{\theta(\gamma + bc)(\alpha + bc)}{c(\gamma - \alpha)}\right)^{2}\chi^{2}.$$

Comparison

Comparing cooperation and non-cooperation.

The difference for inflation between cooperation and non-cooperation is given by the following equation:

$$\pi_{C}^{*} - \pi_{n}^{*} = \left[\frac{\theta c(\alpha + \beta + 2bc)}{\left[\frac{\theta}{2}(\alpha + \beta + 2bc)(\alpha + \beta) + 2c^{2}\right]} - \frac{2c\theta(\alpha + bc)}{\theta(\alpha + bc)(\alpha + \beta) + 2c^{2}}\right]\chi$$

$$-\left[\frac{\frac{\theta}{2}c(\alpha + \beta + 2bc)}{\frac{\theta}{2}(\alpha + \beta + 2bc)^{2} + 2c^{2}} - \frac{\theta c(\alpha + bc)}{\theta(\alpha + bc)(\alpha + \beta) + 2c^{2}}\right](u_{1} + u_{2})$$

$$+\left[\frac{\left[\frac{\theta}{2}(\alpha + \beta + 2\mathrm{bc})\right]}{\left[\frac{\theta}{2}(\alpha + \beta + 2\mathrm{bc})^{2} + 2\mathrm{c}^{2}\right]} - \frac{\theta(\alpha + \mathrm{bc})}{\theta(\alpha + \mathrm{bc})(\alpha + \beta) + 2\mathrm{c}^{2}}\right](\alpha + \beta)\varepsilon$$

$$= \left[\frac{\theta c(\alpha + \beta + 2bc)\left[\theta(\alpha + bc)(\alpha + \beta) + 2c^{2}\right] - 2c\theta(\alpha + bc)\left[\frac{\theta}{2}(\alpha + \beta + 2bc)(\alpha + \beta) + 2c^{2}\right]}{\left[\theta(\alpha + bc)(\alpha + \beta) + 2c^{2}\right]}\right]\chi$$

$$-\left[\frac{\frac{\theta}{2}c(\alpha + \beta + 2bc)\left[\theta(\alpha + bc)(\alpha + \beta) + 2c^{2}\right] - \left[\frac{\theta}{2}(\alpha + \beta + 2bc)^{2} + 2c^{2}\right]\theta c(\alpha + bc)}{\left[\frac{\theta}{2}(\alpha + \beta + 2bc)^{2} + 2c^{2}\right]}\right](u_{1} + u_{2})$$

$$+\left[\frac{\left[\frac{\theta}{2}\left(\alpha+\beta+2\mathrm{bc}\right)\right]\left[\theta\left(\alpha+\mathrm{bc}\right)\left(\alpha+\beta\right)+2\mathrm{c}^2\right]-\theta\left(\alpha+\mathrm{bc}\right)\left[\frac{\theta}{2}\left(\alpha+\beta+2\mathrm{bc}\right)^2+2\mathrm{c}^2\right]}{\left[\frac{\theta}{2}\left(\alpha+\beta+2\mathrm{bc}\right)^2+2\mathrm{c}^2\right]\left[\theta\left(\alpha+\mathrm{bc}\right)\left(\alpha+\beta\right)+2\mathrm{c}^2\right]}\right]\left(\alpha+\beta\right)\varepsilon.$$

All terms in brackets are negative for $\alpha > \beta$. In particular the inflation bias (the constant term) is weaker and the variance of inflation smaller in cooperation than in non-cooperation.

The difference for national fiscal authority between cooperation and non-cooperation is given by the following equation:

$$y_C^* - y_{in}^* = \left[\frac{\theta\left(\alpha+bc\right)\left(\alpha+\beta\right)}{\theta\left(\alpha+bc\right)\left(\alpha+\beta\right)+2c^2} - \frac{\theta\left(\alpha+bc\right)\left(\alpha+\beta\right)}{\theta\left(\alpha+bc\right)\left(\alpha+\beta\right)+2c^2}\right]\chi$$

$$-\left[\frac{1}{\left[\frac{\theta}{2}\left(\alpha+\beta+2bc\right)^2+2c^2\right]} - \frac{1}{\theta\left(\alpha+bc\right)\left(\alpha+\beta+2bc\right)+2c^2}\right]c^2\left(u_1+u_2\right)$$

$$+\left[\frac{1}{\left[\frac{\theta}{2}\left(\alpha+\beta+2bc\right)^2+2c^2\right]} - \frac{1}{\theta\left(\alpha+bc\right)\left(\alpha+\beta+2bc\right)+2c^2}\right]\left(\alpha+\beta\right)c\varepsilon$$

$$= -\left[\frac{\frac{\theta}{2}\left(\alpha+\beta+2bc\right)\left[\alpha-\beta\right]}{\left[\frac{\theta}{2}\left(\alpha+\beta+2bc\right)^2+2c^2\right]\left[\theta\left(\alpha+bc\right)\left(\alpha+\beta+2bc\right)+2c^2\right]}\right]\left(c^2\left(u_1+u_2\right)-\left(\alpha+\beta\right)c\varepsilon\right).$$

We notice that the term in brackets for $\alpha > \beta$. Thus the variance of output is weaker in cooperation than in non-cooperation, as expected.

Comparing cooperation and federalism.

The difference for inflation between cooperation and federalism is given by the following equation:

$$\pi_C^* - \pi_F^* = \left[\frac{\theta c\left(\alpha+\beta+2bc\right)}{\left[\frac{\theta}{2}\left(\alpha+\beta+2bc\right)\left(\alpha+\beta\right)+2c^2\right]} - \frac{\theta\left(\alpha+bc\right)\left(\gamma+bc\right)}{c\left(\gamma-\alpha\right)}\right]\chi$$

$$-\left[\frac{\frac{\theta}{2}c\left(\alpha+\beta+2bc\right)}{\frac{\theta}{2}\left(\alpha+\beta+2bc\right)^2+2c^2}\right]\left(u_1+u_2\right)+\left[\frac{\left[\frac{\theta}{2}\left(\alpha+\beta+2\mathrm{bc}\right)\right]\left(\alpha+\beta\right)}{\left[\frac{\theta}{2}\left(\alpha+\beta+2\mathrm{bc}\right)^2+2\mathrm{c}^2\right]}\right]\left(\alpha+\beta\right)\varepsilon$$

$$= \left[\frac{\theta c\left(\alpha+\beta+2bc\right)c\left(\gamma-\alpha\right)-\theta\left(\alpha+bc\right)\left(\gamma+bc\right)\left[\frac{\theta}{2}\left(\alpha+\beta+2bc\right)\left(\alpha+\beta\right)+2c^2\right]}{\left[\frac{\theta}{2}\left(\alpha+\beta+2bc\right)\left(\alpha+\beta\right)+2c^2\right]c\left(\gamma-\alpha\right)}\right]\chi$$

$$-\left[\frac{\frac{\theta}{2}c\left(\alpha+\beta+2bc\right)}{\frac{\theta}{2}\left(\alpha+\beta+2bc\right)^2+2c^2}\right]\left(u_1+u_2\right)+\left[\frac{\left[\frac{\theta}{2}\left(\alpha+\beta+2\mathrm{bc}\right)\right]\left(\alpha+\beta\right)}{\left[\frac{\theta}{2}\left(\alpha+\beta+2\mathrm{bc}\right)^2+2\mathrm{c}^2\right]}\right]\left(\alpha+\beta\right)\varepsilon.$$

The first term in brackets is positive for $\alpha > \gamma$, as well as the two others. Thus the inflation bias is higher in cooperation than in a federation and the variance of inflation is higher.

The difference of national aggregate products is:

$$y_{iC}^* - y_{iF}^* = \left[\frac{\theta\left(\alpha+bc\right)\left(\alpha+\beta\right)}{\theta\left(\alpha+bc\right)\left(\alpha+\beta\right)+2c^2} - \frac{\left(\alpha+bc\right)}{\left(\gamma-\alpha\right)}\right]\chi + \frac{\left(\alpha+\beta\right)c}{\left[\frac{\theta}{2}\left(\alpha+\beta+2bc\right)^2+2c^2\right]}\varepsilon$$

$$-\left[\frac{c^2}{\left[\frac{\theta}{2}\left(\alpha+\beta+2bc\right)^2+2c^2\right]} - \frac{2\gamma\beta}{\left(2\gamma-\alpha-\beta\right)\left(\alpha-\beta\right)}\right]u_i + \frac{2\gamma\beta c^2}{\left(2\gamma-\alpha-\beta\right)\left(\alpha-\beta\right)}u_j$$

$$= \left[\frac{(\alpha+bc)\left[\theta(\alpha+\beta)(\gamma-2\alpha-bc)-2c^2\right]}{(\gamma-\alpha)}\right]\chi + \frac{(\alpha+\beta)c}{\left[\frac{\theta}{2}(\alpha+\beta+2bc)^2+2c^2\right]}\varepsilon$$

$$-\left[\frac{c^2(2\gamma-\alpha-\beta)(\alpha-\beta)-\theta\gamma\beta(\alpha+\beta+2bc)^2-4\gamma\beta c^2}{\left[\frac{\theta}{2}(\alpha+\beta+2bc)^2+2c^2\right](2\gamma-\alpha-\beta)(\alpha-\beta)}\right]u_i + \frac{2\gamma\beta}{(2\gamma-\alpha-\beta)(\alpha-\beta)}c^2 u_j.$$

The sign of first term in brackets is ambiguous even when $\alpha > \gamma$, as well as the third one. The two other terms are postive when $\alpha > \beta$. Thus this difference is either positive or negative.

For the union's aggregate product, the difference is:

$$y_C^* - \overline{y}_F^* = \left[\frac{\theta(\alpha+bc)(\alpha+\beta)}{\theta(\alpha+bc)(\alpha+\beta)+2c^2} - \frac{(\alpha+bc)}{(\gamma-\alpha)}\right]\chi$$

$$-\left[\frac{c^2}{\left[\frac{\theta}{2}(\alpha+\beta+2bc)^2+2c^2\right]} - \frac{\gamma\beta}{(2\gamma-\alpha-\beta)(\alpha-\beta)}\right](u_1+u_2) + \frac{1}{\left[\frac{\theta}{2}(\alpha+\beta+2bc)^2+2c^2\right]}(\alpha+\beta)c\varepsilon$$

$$= \frac{(\alpha+bc)\left(\theta(\alpha+\beta)(\gamma-2\alpha-bc)-2c^2\right)}{\left(\theta(\alpha+bc)(\alpha+\beta)+2c^2\right)(\gamma-\alpha)}\chi$$

$$-\frac{c^2(2\gamma-\alpha-\beta)(\alpha-\beta)-\gamma\beta\frac{\theta}{2}(\alpha+\beta+2bc)^2-2\gamma\beta c^2}{\left[\frac{\theta}{2}(\alpha+\beta+2bc)^2+2c^2\right](2\gamma-\alpha-\beta)(\alpha-\beta)}(u_1+u_2) + \frac{1}{\left[\frac{\theta}{2}(\alpha+\beta+2bc)^2+2c^2\right]}(\alpha+\beta)c\varepsilon.$$

For similar reasons as above, this difference is either positive or negative even though the first and the third terms are positive.

Impact of fiscal spillovers on expected losses.

$$B_{iF}^* = \theta\frac{2\gamma^2\beta^2}{(2\gamma-\alpha-\beta)^2(\alpha-\beta)^2}$$

$$\frac{\partial B_{iF}^*}{\partial\gamma} = \theta\frac{4\gamma\beta^2(2\gamma-\alpha-\beta)^2(\alpha-\beta)^2-2\gamma^2\beta^2.4(2\gamma-\alpha-\beta)(\alpha-\beta)^2}{\left((2\gamma-\alpha-\beta)^2(\alpha-\beta)^2\right)^2}$$

$$= -\frac{4\theta\gamma\beta^2(\alpha+\beta)}{(2\gamma-\alpha-\beta)^2(\alpha-\beta)^2} < 0.$$

$$\frac{\partial B_{iF}^*}{\partial\beta} = \theta\frac{4\gamma^2\beta(2\gamma-\alpha-\beta)^2(\alpha-\beta)^2}{\left((2\gamma-\alpha-\beta)^2(\alpha-\beta)^2\right)^2} + 2\theta\frac{\left(2\gamma^2\beta^2\right)(\alpha-\beta)(2\gamma-\alpha-\beta)\left((\alpha-\beta)+(2\gamma-\alpha-\beta)\right)}{\left((2\gamma-\alpha-\beta)^2(\alpha-\beta)^2\right)^2}$$

$$= 4\gamma^2\beta\theta\left(\frac{2\gamma\alpha-\alpha^2-\beta^2}{(2\gamma-\alpha-\beta)^2(\alpha-\beta)^2}\right) = -4\gamma^2\beta\theta\left(\frac{\alpha^2+\beta^2-2\gamma\alpha}{(2\gamma-\alpha-\beta)^2(\alpha-\beta)^2}\right) < 0.$$

$$A_{iF}^* = \frac{1}{2}\frac{\theta(\gamma+bc)^2\left(c^2+\theta(\alpha+bc)^2\right)}{c^2(\gamma-\alpha)^2}$$

$$\frac{\partial A_{iF}^*}{\partial\gamma} = \frac{1}{2}\frac{2\theta(\gamma+bc)\left(c^2+\theta(\alpha+bc)^2\right)c^2(\gamma-\alpha)^2-2\theta(\gamma+bc)^2\left(c^2+\theta(\alpha+bc)^2\right)c^2(\gamma-\alpha)}{\left(c^2(\gamma-\alpha)^2\right)^2}$$

$$= -\frac{\theta\left(\gamma+bc\right)\left(c^2+\theta\left(\alpha+bc\right)^2\right)\left(\gamma-\alpha\right)c^2\left(\alpha+bc\right)}{\left(c^2\left(\gamma-\alpha\right)^2\right)^2} = \frac{\theta\left(\gamma+bc\right)\left(c^2+\theta\left(\alpha+bc\right)^2\right)\left(\alpha+bc\right)}{c^2\left(\alpha-\gamma\right)^3} > 0$$

and

$$\frac{\partial A^*_{iF}}{\partial\beta} = 0.$$

$$A^*_{iC} = \frac{1}{2}\theta\frac{\left(\frac{\theta}{2}\left(\alpha+\beta+2bc\right)\left(\alpha+\beta\right)-2c^2\right)^2+\theta c^2\left(\alpha+\beta+2bc\right)^2}{\left(\frac{\theta}{2}\left(\alpha+\beta+2bc\right)\left(\alpha+\beta\right)+2c^2\right)^2}$$

$$\frac{\partial A^*_{iC}}{\partial\beta} = \theta\frac{\left(\frac{\theta}{2}\left(\alpha+\beta+2bc\right)\left(\alpha+\beta\right)-2c^2\right)\left(\frac{\theta}{2}\left(\alpha+\beta\right)+\frac{\theta}{2}\left(\alpha+\beta+2bc\right)\right)+\theta c^2\left(\alpha+\beta+2bc\right)}{\left(\frac{\theta}{2}\left(\alpha+\beta+2bc\right)\left(\alpha+\beta\right)+2c^2\right)^2}$$

$$-\theta\frac{\left(\left(\frac{\theta}{2}\left(\alpha+\beta+2bc\right)\left(\alpha+\beta\right)-2c^2\right)^2+\theta c^2\left(\alpha+\beta+2bc\right)^2\right)\left(\frac{\theta}{2}\left(\alpha+\beta\right)+\frac{\theta}{2}\left(\alpha+\beta+2bc\right)\right)}{\left(\frac{\theta}{2}\left(\alpha+\beta+2bc\right)\left(\alpha+\beta\right)+2c^2\right)^3}.$$

After tedious computations we eventually get:

$$\frac{\partial A^*_{iC}}{\partial\beta} = \theta^2\frac{\left(\alpha+\beta\right)\left(\frac{\theta}{2}\left(\alpha+\beta+2bc\right)\left(\alpha+\beta+bc\right)-c^2\right)}{\left(\frac{\theta}{2}\left(\alpha+\beta+2bc\right)\left(\alpha+\beta\right)+2c^2\right)^2}$$

$$-\theta^2\frac{\left(\left(\frac{\theta}{2}\left(\alpha+\beta+2bc\right)\left(\alpha+\beta\right)-2c^2\right)^2+\theta c^2\left(\alpha+\beta+2bc\right)^2\right)\left(\frac{\theta}{2}\left(\alpha+\beta+2bc\right)\left(\alpha+\beta\right)+2c^2\right)\left(\alpha+\beta+bc\right)}{\left(\frac{\theta}{2}\left(\alpha+\beta+2bc\right)\left(\alpha+\beta\right)+2c^2\right)^4}$$

which is either positive or negative.

$$B^*_{iC} = \frac{c^2\left(\left(\frac{\theta}{2}\right)^2\left(\alpha+\beta+2bc\right)^2+\theta c^2\right)}{\left(\frac{\theta}{2}\left(\alpha+\beta+2bc\right)^2+2c^2\right)^2}$$

$$\frac{\partial B^*_{iC}}{\partial\beta} = \frac{c^2\left(\left(\frac{\theta}{2}\right)^2 2\left(\alpha+\beta+2bc\right)\right)\left(\frac{\theta}{2}\left(\alpha+\beta+2bc\right)^2+2c^2\right)^2 - c^2\left(\left(\frac{\theta}{2}\right)^2\left(\alpha+\beta+2bc\right)^2+\theta c^2\right)2\left(\left(\frac{\theta}{2}\left(\alpha+\beta+2bc\right)^2+2c^2\right.\right.}{\left(\frac{\theta}{2}\left(\alpha+\beta+2bc\right)^2+2c^2\right)^4}$$

$$= \frac{2c^2\left(\frac{\theta}{2}\right)^2\left(\alpha+\beta+2bc\right)\left(1-\left(\alpha+\beta+2bc\right)\right)}{\left(\frac{\theta}{2}\left(\alpha+\beta+2bc\right)^2+2c^2\right)^2} \lesseqgtr 0.$$

$$C^*_{iC} = \frac{1}{2}\frac{\left(\left(\frac{\theta}{2}\right)^2\left(\alpha+\beta+2bc\right)^2+\theta c^2\right)\left(\alpha+\beta\right)}{\left(\frac{\theta}{2}\left(\alpha+\beta+2bc\right)^2+2c^2\right)^2} = B^*_{iC}\left(\alpha+\beta\right)$$

$$\frac{\partial C^*_{iC}}{\partial\beta} = \frac{\partial B^*_{iC}}{\partial\beta}\left(\alpha+\beta\right)+B^*_{iC} = \frac{2c^2\left(\frac{\theta}{2}\right)^2\left(\alpha+\beta+2bc\right)\left(1-\left(\alpha+\beta+2bc\right)\right)}{\left(\frac{\theta}{2}\left(\alpha+\beta+2bc\right)^2+2c^2\right)^2}\left(\alpha+\beta\right)+\frac{c^2\left(\left(\frac{\theta}{2}\right)^2\left(\alpha+\beta+2bc\right)^2+\theta c^2\right)}{\left(\frac{\theta}{2}\left(\alpha+\beta+2bc\right)^2+2c^2\right)^2}$$

$$= \frac{c^2\left(\left(\frac{\theta}{2}\right)^2(\alpha+\beta+2bc)\left(2+(\alpha+\beta+2bc)\left(1-2(\alpha+\beta)\right)\right)+\theta c^2\right)}{\left(\frac{\theta}{2}(\alpha+\beta+2bc)^2+2c^2\right)^2} \gtreqless 0.$$

Comparing expected losses in cooperation and in federation.

Given the equations of expected losses for national fiscal authorities in the three configurations, we deduce the expected losses for national fiscal authorities in the three configurations of interest:

$$E\left(L_{in}^*\right) = \frac{2\theta c^2\left(c^2+\theta(\alpha+bc)^2\right)}{\left(\theta(\alpha+bc)(\alpha+\beta)+2c^2\right)^2}\chi^2$$

$$+\frac{\theta c^2\left(c^2+\theta(\alpha+bc)^2\right)}{\left(\theta(\alpha+bc)(\alpha+\beta+2bc)+2c^2\right)^2}\sigma_u^2+\frac{1}{2}\frac{\theta c^2\left((\alpha+\beta)^2+\theta(\alpha+bc)^2\right)}{\left(\theta(\alpha+bc)(\alpha+\beta+2bc)+2c^2\right)^2}\sigma_\varepsilon^2$$

$$E\left(L_{iF}^*\right) = \frac{1}{2}\frac{\theta(\gamma+bc)^2\left(c^2+\theta(\alpha+bc)^2\right)}{c^2(\gamma-\alpha)^2}\chi^2+\theta\frac{2\gamma^2\beta^2}{(2\gamma-\alpha-\beta)^2(\alpha-\beta)^2}\sigma_u^2$$

and

$$E\left(L_{iC}^*\right) = \frac{1}{2}\theta\frac{\left(\frac{\theta}{2}(\alpha+\beta+2bc)(\alpha+\beta)-2c^2\right)^2+\theta c^2(\alpha+\beta+2bc)^2}{\left(\frac{\theta}{2}(\alpha+\beta+2bc)(\alpha+\beta)+2c^2\right)^2}\chi^2$$

$$+\frac{c^2\left(\left(\frac{\theta}{2}\right)^2(\alpha+\beta+2bc)^2+\theta c^2\right)}{\left(\frac{\theta}{2}(\alpha+\beta+2bc)^2+2c^2\right)^2}\sigma_u^2+\frac{1}{2}\frac{\left(\left(\frac{\theta}{2}\right)^2(\alpha+\beta+2bc)^2+\theta c^2\right)(\alpha+\beta)}{\left(\frac{\theta}{2}(\alpha+\beta+2bc)^2+2c^2\right)^2}\sigma_\varepsilon^2.$$

All terms are positive.

$$B_{iF}^*-B_{iC}^* = \theta\frac{2\gamma^2\beta^2}{(2\gamma-\alpha-\beta)^2(\alpha-\beta)^2}-\frac{c^2\left(\left(\frac{\theta}{2}\right)^2(\alpha+\beta+2bc)^2+\theta c^2\right)}{\left(\frac{\theta}{2}(\alpha+\beta+2bc)^2+2c^2\right)^2}$$

$$= \frac{2\theta\gamma^2\beta^2\left(\frac{\theta}{2}(\alpha+\beta+2bc)^2+2c^2\right)^2-c^2\frac{\theta}{2}\left(\frac{\theta}{2}(\alpha+\beta+2bc)^2+2c^2\right)(2\gamma-\alpha-\beta)^2(\alpha-\beta)^2}{(2\gamma-\alpha-\beta)^2(\alpha-\beta)^2\left(\frac{\theta}{2}(\alpha+\beta+2bc)^2+2c^2\right)^2}$$

$$= \frac{2\theta\gamma^2\beta^2\left(\frac{\theta}{2}(\alpha+\beta+2bc)^2+2c^2\right)-c^2\frac{\theta}{2}(2\gamma-\alpha-\beta)^2(\alpha-\beta)^2}{(2\gamma-\alpha-\beta)^2(\alpha-\beta)^2\left(\frac{\theta}{2}(\alpha+\beta+2bc)^2+2c^2\right)^2}$$

$$= \frac{2\theta\gamma^2\beta^2}{(2\gamma-\alpha-\beta)^2(\alpha-\beta)^2\left(\frac{\theta}{2}(\alpha+\beta+2bc)^2+2c^2\right)}-\frac{c^2\frac{\theta}{2}}{\left(\frac{\theta}{2}(\alpha+\beta+2bc)^2+2c^2\right)^2}.$$

It is negative if the spillover parameters (β and γ) are small and positive if they are large and close to α.

$$A_{iF}^*-A_{iC}^* = \frac{1}{2}\frac{\theta(\gamma+bc)^2\left(c^2+\theta(\alpha+bc)^2\right)}{c^2(\gamma-\alpha)^2}-\frac{1}{2}\theta\frac{\left(\frac{\theta}{2}(\alpha+\beta+2bc)(\alpha+\beta)-2c^2\right)^2+\theta c^2(\alpha+\beta+2bc)^2}{\left(\frac{\theta}{2}(\alpha+\beta+2bc)(\alpha+\beta)+2c^2\right)^2}$$

$$= \frac{1}{2}\theta \left[\frac{(\gamma + bc)^2 \left(c^2 + \theta (\alpha + bc)^2\right)}{c^2 (\gamma - \alpha)^2} - \frac{\left(\frac{\theta}{2} (\alpha + \beta + 2bc) (\alpha + \beta) - 2c^2\right)^2 + \theta c^2 (\alpha + \beta + 2bc)^2}{\left(\frac{\theta}{2} (\alpha + \beta + 2bc) (\alpha + \beta) + 2c^2\right)^2} \right]$$

$$= \frac{1}{2}\theta \left[\frac{(\gamma + bc)^2 \left(\theta (\alpha + bc)^2\right)}{c^2 (\gamma - \alpha)^2 \left(\frac{\theta}{2} (\alpha + \beta + 2bc) (\alpha + \beta) + 2c^2\right)^2} \right]$$

$$+ \frac{1}{2}\theta c^2 \left[\frac{(\gamma + bc)^2 - (\gamma - \alpha)^2 \left(\left(\frac{\theta}{2} (\alpha + \beta + 2bc) (\alpha + \beta) - 2c^2\right)^2 + \theta c^2 (\alpha + \beta + 2bc)^2 \right)}{c^2 (\gamma - \alpha)^2 \left(\frac{\theta}{2} (\alpha + \beta + 2bc) (\alpha + \beta) + 2c^2\right)^2} \right].$$

It is positive if γ is sufficiently close to α.